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## MULTIPOINT PROBLEM FOR PSEUDODIFFERENTIAL EQUATIONS

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**Abstract.** Correctness of the problem with multipoint conditions in time variable and frequency of the spatial coordinates for the partial differential, equations, not solved with respect to the highest derivative, with pseudodifferential operators is investigated. The conditions of existence and uniqueness of the problem solution, metric theorems on lower bounds of small denominators arising in the construction of the solution of the problem are proved.

**Keywords:** differential equations; multipoint conditions; pseudodifferential operators; small denominators

### 1. Introduction

Problems with multipoint conditions for partial differential equations are, in general, conditionally correct [1-4], and their solvability in many cases is related to the problem of small denominators.

This paper examines the problem with multipoint in time variable conditions for partial differential equations, what are not solved with respect to the highest derivative in time, with pseudodifferential operators.

### 2. Statement of the problem

In the region  $\bar{D} = \{(t, x) : t \in (0, T), x \in \Omega_p\}$ ,  $\Omega_p$  is  $p$ -dimensional torus  $(R/2\pi Z)^p$ , let us consider the problem:

$$A\left(\frac{\partial}{\partial t}, D\right)u \equiv A_n(D)\frac{\partial^n u(t, x)}{\partial t^n} + \quad (1)$$

$$+ \sum_{\beta=0}^{n-1} \alpha_\beta A_\beta(D)\frac{\partial^\beta u(t, x)}{\partial t^\beta} = 0,$$

$$u(t_q, x) = \varphi_q(x), \quad (2)$$

$$q = 1, \dots, n, 0 \leq t_1 < t_2 < \dots < t_n \leq T < \infty,$$

where

$$\alpha_\beta \in C, \beta = 0, 1, \dots, n-1;$$

$$\operatorname{Re} \alpha_0 \neq 0; D = (D_1, \dots, D_p), D_j = \frac{1}{i} \frac{\partial}{\partial x_j}, \quad j = 1, \dots, p;$$

$A_\beta(D), \beta = 0, 1, \dots, n$ , are pseudodifferential operators, whose symbols  $A_\beta(\xi), \beta = 0, 1, \dots, n$ , are real-valued functions,  $k \in \mathbb{Z}^p$ .

Below we use the notations:

$$x = (x_1, \dots, x_p) \in R^p, \quad k = (k_1, \dots, k_p) \in Z^p,$$

$$(k, x) = k_1 x_1 + \dots + k_p x_p, \quad |k| = |k_1| + \dots + |k_p|, \quad \Gamma'$$

is the space of all continuous antilinear functionals over  $\Gamma'$ , which coincides with the space of formal trigonometric series;  $C^n([0, T], \Gamma)$  ( $C^n([0, T], \Gamma')$ ) is the space of functions  $z(t, x)$  defined in the region  $\bar{D}$ ,  $n$  times continuously differentiable in  $t$ , such that for each fixed number  $t \in [0, T]$ ,  $\partial^m z / \partial t^m \in \Gamma(\Gamma')$ ,  $m = 0, 1, \dots, n$ .

The action of the operator  $A_\beta(D), \beta = 0, 1, \dots, n$ , on a periodic function of the form

$$\psi(x) = \sum_{|k| \geq 0} \psi_k \exp(i(k, x)) \quad (3)$$

is define as follows:

$$A_\beta(D)\psi(x) = \sum_{|k| \geq 0} A_\beta(k) \psi_k \exp(i(k, x)),$$

$$\beta = 0, 1, \dots, n. \quad (4)$$

Further, we assume that

$$(\forall k \in Z^p) A_n(k) \neq 0. \quad (5)$$

The solution of the problem (1), (2) we find in the form of a series

$$u(t, x) = \sum_{|k| \geq 0} u_k(t) \exp(i(k, x)). \quad (6)$$

Each of the coefficients  $u_k(t), k \in Z^p$ , is, respectively, the solution of the problem:

$$A_n(k) \frac{d^n u_k(t)}{dt^n} + \sum_{\beta=0}^{n-1} \alpha_\beta A_\beta(k) \frac{d^\beta u_k(t)}{dt^\beta} = 0, \quad (7)$$

$$u_k(t_q) = \varphi_{qk}, \quad q = 1, \dots, n, \quad (8)$$

where  $\varphi_{qk}$  are Fourier coefficients of the function  $\varphi_q(x), q = 1, \dots, n$ .

We denote by  $\lambda_j(k), j = 1, \dots, n$ , the roots of the characteristic equation

$$A_n(\lambda, k) \equiv A_n(k) \lambda^n + \sum_{\beta=0}^{n-1} \alpha_\beta A_\beta(k) \lambda^\beta = 0, \quad (9)$$

which corresponds to the equation (7). We assume that for each  $k \in Z^p, \lambda_j(k), j = 1, \dots, n$ , are distinct and different from zero.

Solution of the problem (7), (8) is represented by the formula

$$u_k(t) = \sum_{j=1}^n C_{kj} \exp(\lambda_j(k)t), \quad (10)$$

where  $C_{kj}, j = 1, \dots, n$ , is a solution of the corresponding system of linear algebraic equations

$$\sum_{j=1}^n C_{kj} \exp(\lambda_j(k)t_q) = \varphi_{qk}, \quad q = 1, \dots, n,$$

which determinant  $\Delta(k)$  has the form

$$\Delta(k) \equiv \det \left\| \exp(\lambda_j(k)t_q) \right\|_{q,j=1}^n. \quad (11)$$

### 3. Uniqueness and existence of the solution.

Problem (1) - (2) can not have two different solutions if and only if the corresponding homogeneous problem has only the trivial solution.

**Theorem 1** For the uniqueness of the solution of the problem (1), (2) in the space  $(C^n([0, T], \Gamma'))$  it is necessary and sufficient that:

$$(\forall k \in Z^p) \Delta(k) \neq 0. \quad (12)$$

**The proof** is carried out according to the theorem 1 proving scheme [1] and follows from the uniqueness of the expansion of functions from  $\Gamma'$  into Fourier series.

Further, we assume that condition (12) is satisfied. Then from (6) and (10) we obtain the formal presentation of the solution of the problem (1), (2) as a series

$$u(t, x) = \sum_{|k| \geq 0} \sum_{j,q=1}^n \frac{\Delta_{qj}(k) \varphi_{qk} \exp((\lambda_j k)t + i(k, x))}{\Delta(k)}, \quad (13)$$

where  $\Delta_{qj}(k)$  – cofactor of the element  $\exp(\lambda_j(k)t_q)$  of the determinant (11).

From the formula (13), taking into consideration that in the space  $\Gamma'$  an arbitrary trigonometric series is convergent we obtain the following statement.

**Theorem 2.** Let the condition (12) is valid. If function  $\varphi_q \in \Gamma(\Gamma'), q = 1, \dots, n$ , then there exists a solution of the problem (1), (2), which belongs to the space  $C^n([0, T], \Gamma) (C^n([0, T], \Gamma'))$ .

Consider the particular case of the problem (1),(2) when

$$A_\beta(D) \equiv (1 - \Delta)^m = (1 + D_1^2 + \dots + D_p^2)^{m\beta}, \quad (14)$$

$$m_\beta \in R, \beta = 0, 1, \dots, n,$$

where  $m_\beta > m_n, \beta = 0, 1, \dots, n-1$ , and conditions (2) fix the status of the process, which is described by equation (1) at regular intervals, i.e.

$$t_q = (q-1)t_0, \quad q = 1, \dots, n, \quad t_0 = T/(n-1). \quad (15)$$

Obviously, the operators' symbols satisfy the conditions

$$\begin{aligned} (\forall k \in Z^p) N_0(1 + |k|^2)^{m\beta} &\leq \\ &\leq |A_\beta(k)| \leq N_1(1 + |k|^2)^{m\beta}, \end{aligned} \quad (16)$$

$$\beta = 0, 1, \dots, n, \quad 0 < N_0 \leq N_1.$$

The characteristic determinant of the problem

(1),(2), (14), (15) is calculated by the formula

$$\Delta(k) = \prod_{1 \leq p < r \leq n} (\exp(\lambda_r(k)t_0) - \exp(\lambda_p(k)t_0)). \quad (17)$$

Theorem 1 and formula (17) implies that for uniqueness of the solution of the considered problem it is necessary and sufficient that

$$\begin{aligned} (\lambda_p(k) - \lambda_r(k))t_0 &\neq 2\pi il, \\ l \in Z, k \in Z^p, 1 \leq p < r \leq n. \end{aligned} \quad (18)$$

By conditions (18) the solution of problem (1), (2),(14), (15) is formally represented as a series

$$\begin{aligned} u(t, x) = \\ = \sum_{|k| \geq 0} \sum_{j,q=1}^n \frac{(-1)^{q-1} \varphi_{qk} S_{n-q}^{(j)} \exp(\lambda_j(k)t + i(k, x))}{\prod_{\substack{p=1 \\ p \neq j}}^n \exp(\lambda_p(k)t_0) - \exp(\lambda_p(k)t_0)}, \end{aligned} \quad (19)$$

where  $S_{(n-q)}^{(j)}$  is the sum of the products of elements  $\exp(\lambda_j(k)t_0)$ ,  $r = 1, \dots, n$ ,  $r \neq j$ ,  $j = 1, \dots, n$ , taken for  $(n-q)$  in each product,  $S_0^{(j)} \equiv 1$ ,  $j = 1, \dots, n$ .

Let us consider case  $m < m_n$ , where  $m = \max_{0 \leq \beta \leq n-1} \{m_\beta\}$ .

In this case, the roots  $\lambda_j(k)$ ,  $j = 1, \dots, n$ , satisfy the inequalities

$$|\lambda_j(k)| \leq H_1 (1 + |k|^2)^\gamma, \quad j = 1, \dots, n. \tag{21}$$

where  $\gamma = -(m - m_n)/n$ ,  $H_1 > 0$  is a positive constant, which does not depend on  $k \in Z^p$ .

**Theorem 3** Let condition (12) is valid and there exists positive constants such that for all (except a finite number of) vectors  $k \in Z^p$ , the inequality

$$|\Delta(k)| > |k|^{-\beta_1}, \tag{22}$$

is satisfied.

If  $\varphi_q \in C^h(\Omega)$ ,  $h = [n\gamma + \beta_1 + m_n + p/2] + 1$ ,  $q = 1, \dots, n$ , then there exists a solution of the problem (1), (2) from the space  $C^{(n, m_n)}(\bar{D})$ , that continuously depends on functions  $\varphi_q(x)$ ,  $q = 1, \dots, n$ .

**The proof.** If functions  $\varphi_q(x)$ ,  $q = 1, \dots, n$  satisfy the conditions of the theorem, then the inequalities

$$|\varphi_{qk}| \leq \tilde{N}_1 |k|^{-h} \|\varphi_q(x)\|_{C^h(\Omega)}, C_1 > 0. \tag{23}$$

are hold true.

From the estimates (21) (23) and formula (13) for norms of the solution of the problem (1), (2) we have the estimate

$$\begin{aligned} \|u\|_{C^{(n, m_n)}(\bar{D})} &\leq C_2 \sum_{|k|>0} |k|^{-p-\varepsilon} \sum_{q=1}^n \|\varphi_q\|_{C^h(\Omega)} \leq \\ &\leq C_3 \sum_{q=1}^n \|\varphi_q\|_{C^h(\Omega)}. \end{aligned} \tag{24}$$

From the inequality (24) follows the proving of the theorem.

#### 4. Metrical estimations

**Lemma 1.** Let condition (12) and evaluations (21) are fulfilled. Then for any fixed  $\text{Im } \alpha_0, \alpha_1, \dots, \alpha_{n-1}$  and almost all (with respect to Lebesgue measure in  $R^p$ ) numbers  $\text{Re } \alpha_0$  inequality holds

$$\prod_{1 \leq j < r \leq n} |\lambda_r(k) - \lambda_j(k)| > M_1 (1 + |k|^2)^{-(m_n - m_0 + p/2)/(n-1)/2}, \tag{25}$$

$$M_1 > 0,$$

for all (except for a finite number) vectors  $k \in Z^p$ .

**Theorem 4.** For an arbitrary fixed  $\text{Im } \alpha_0, \alpha_1, \dots, \alpha_{n-1}$  and almost all (concerning the Lebesgue measure  $R^{n+1}$ ) vectors  $(\text{Re } \alpha_0, \bar{t})$ , where  $\bar{t} = (t_1, \dots, t_n)$ , inequality (22) holds for all (except for a finite number of) vectors  $k \in Z^p$  and  $\beta_1 > (n-1)/2((n+1)p/2 + m - m_0)$ .

**The proving** is based on lemma 1 and carried out the scheme of proving of the theorem 3 [1].

Let us consider case  $m = m_n$ .

Estimates (21) in this case have the form

$$|\lambda_q(k)| \leq H_3, H_3 > 0, k \in Z^p, q = 1, \dots, n. \tag{26}$$

**Theorem 5.** Let  $m = m_n$ , and conditions (26) are hold true. If  $\varphi_q \in C^{h_1}(\Omega)$ ,  $q = 1, \dots, n$ ,  $h_1 = [p/4(n^2 + 1) + (m - m_0)(n-1)/2 + m_n] + 1$ , then for arbitrary fixed  $\text{Im } \alpha_0, \alpha_1, \dots, \alpha_{n-1}$  and almost all (concerning to Lebesgue measure in  $R^{n+1}$ ) vectors  $(\text{Re } \alpha_0, \bar{t})$  there exists the solution of the problem (1), (2) in the space  $C^{(n, m_n)}(\bar{D})$ , which continuously depends on the functions  $\varphi_q(x)$ ,  $q = 1, \dots, n$ .

**The proof.** Note that for almost all (concerning to Lebesgue measure  $R^{n+1}$ ) vectors  $(\text{Re } \alpha_0, \bar{t})$ , will get evaluation

$$|\Delta(k)| > |k|^{-(p/2(n+1) + m_n - m_0)(n-1) - \varepsilon}, \varepsilon > 0. \tag{27}$$

From the estimates (26), (27) and formula (13) for the norm of the solution of the problem we obtain the estimate

$$\begin{aligned} \|u\|_{C^{(n, m_n)}(\bar{D})} &\leq C_4 \sum_{|k|>0} |k|^{-p-\varepsilon} \sum_{q=1}^n \|\varphi_q\|_{C^{h_1}(\Omega)} \leq \\ &\leq C_5 \sum_{q=1}^n \|\varphi_q\|_{C^{h_1}(\Omega)}. \end{aligned} \tag{28}$$

From the inequality (28) follows theorem proving.

#### 5. Conclusion

It was established conditions for the existence, uniqueness and continuous dependence from the right parts of the boundary conditions of the solution

of the multipoint problem for linear equations with pseudodifferential operators.

The results complement scientific works that have been studied in [2-4]. They can be used in the study of specific practice problems which are modeled by means considered in the problem, and in further theoretical studies of problems with multipoint conditions for partial differential equations.

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## **І.С. Ключ<sup>1</sup>, П.Б. Васишин<sup>2</sup>, В.С. Коновалюк<sup>3</sup>. Багатоточкова задача для псевдодиференціальних рівнянь.**

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Досліджено коректність задачі з багатоточковими умовами за часовою змінною та умовами періодичності за просторовими координатами для псевдодиференціальних рівнянь. Встановлено умови існування та єдиності розв'язку задачі. Доведено метричні теореми про оцінки знизу малих знаменників, які виникають при побудові розв'язку задачі.

**Ключові слова:** багатоточкові умови; диференціальні рівняння; малі знаменники; псевдодиференціальні оператори

## **И.С. Ключ<sup>1</sup>, П.Б. Васишин<sup>2</sup>, В.С. Коновалюк<sup>3</sup>. Многоточечная задача для псевдодифференциальных уравнений.**

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Исследована корректность задачи с многоточечными условиями по часовой переменной и периодическими условиями по пространственным координатам для псевдодифференциальных уравнений. Установлены условия существования и единственности решения задачи. Доказаны метрические теоремы об оценках снизу малых знаменателей, которые возникают при построении решения задачи.

**Ключевые слова:** дифференциальные уравнения, малые знаменатели, многоточечные условия; псевдодифференциальные операторы

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