INFORMATION   TECHNOLOGY

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THE SOLUTION OF THE EQUATIONS APPLYING THE GENERAL SERIES OF FIBONACCI

Abstract. *The effective method of solving equations using Fibonacci general series has been shown. The concept of the Fibonacci general series has been applied; the recurrence formula of the population volume change has been obtained. We get the equation based on the recurrence formulas; the largest of this equation is the index of the population volume change. There are some examples of the linear solving equations by means usage the Fibonacci general series in the frame of Excel.*

Keywords: boundary ratio; convergence of the series; equations; Fibonacci series; population; the index of change.

1. Introduction

Numerical methods of the solving equations are considered as widely used in case of scientific and practical problems saving.

The development of effective methods for solving equations has always been an important task up to now.

It has been arised the necessity to introduce the notion of the general series of Fibonacci and the notion of index change when creating of solving equations method using of the Fibonacci general series.

2. Analysis of publications

There are well-known iterative numerical methods to approximate to the equation root [1,2] which have several disadvantages.

For example, the basic idea of Newton's method is as follows: the given interval where the function changes sign is chosen to the initial root approximation. In such case the derivative is found, and it is important to find the approximate equation root using the iterative formula with given error.

Newton's method being a constant, under conditions of chosen point (of the initial approximation) where the function value and its second derivative are the same .

If the initial approximation is not sufficiently close to the solving equation than the method might be divergent.

To solve the equation using the general series of Fibonacci there is not necessity to find the derivative of a function, the convergence of a sequence to the solution of the equation does not depend on the initial values .

Fibonacci series is a series in which each following element is the sum of the previous two since the third one.

It simulates the reproduction of immortal rabbits from the one pair, each pair gives birth only to one pair, and each new pair is appeared rabbits in an exact time.

The series looks like:

$$x_1 = 1, \ x_2 = 1, \ x_3 = x_1 + x_2, \ x_k = x_{k-2} + x_{k-1}, k \geq 3 .$$

There is a condition for the following series [3] :

$$\lim_{k \to \infty} \frac{x_k}{x_{k-1}} = 1,618$$

is the solution of the equation

$$q^2 - q - 1 = 0$$

and equals to the value of the golden ratio

$$q = \frac{1 + \sqrt{5}}{2},$$

which is widely used in the different spheres of science and practice (Table 1).

The given model does not reflect the actual process of population's reproduction.

The research **purpose** is an improvement and effectiveness of solving equations (there is not necessity to find the intervals where the function changes sign and also the function value and its second derivative), finding out the content of this ratio of the current element ratio to the previous

Table 1. The Fibonacci series and the ratio of its elements

Series of Fibonacci	1	1	2	3
The ratio			1,5	1,67
Series of Fibonacci	5	8	21	55
The ratio	1,6	1,625	1,619	1,618
Series of Fibonacci	89	144		
The ratio	1,618	1,618		

element of the general Fibonacci series, modeling volume change of the population.

3. Presentation materials of research

The general series of Fibonacci. Let's consider the most general case.

Rabbits can die at mature age or die because of predators; in case of babies not surviving or any other reasons.

Supposing we have a series x_1, x_2, …, x_n,… in which the recurrent condition is taken from the third element:

$$x_k = \alpha x_{k-2} + \beta x_{k-1}, \quad k \geq 3 .$$

Each next element is equal to the sum of the previous two, which are multiplied respectively by α and β.

We call this series as the general series of Fibonacci.

Upon condition $\alpha = \beta = 1$, we get the Fibonacci series.

And there is a condition for this series, as well as for a series of Fibonacci:

$$\frac{\alpha x_{k-2} + \beta x_{k-1}}{x_{k-1}} = \frac{x_{k-1}}{x_{k-2}} = q .$$

As follows from the proportion, when

$$x_{k-1} = q x_{k-2}$$

the ratio is

$$\frac{\alpha x_{k-2} + q \beta x_{k-2}}{q x_{k-2}} = q$$

we get a square equation:

$$q^2 - \beta q - \alpha = 0 ,$$

where we can find q .

The larger root of the equation we can find using the following formula :

$$q* = \frac{\beta + \sqrt{\beta^2 + 4\alpha}}{2} .$$

Namely

$$\lim_{k \to \infty} \frac{x_k}{x_{k-1}} = \frac{\beta + \sqrt{\beta^2 + 4\alpha}}{2} .$$

If $q* < 1$, than a series of elements of the general Fibonacci series is degressive (decreasing population), and if $q* > 1$, than the series is increasing (population increases).

The speed of ratio convergence $\frac{x_k}{x_{k-1}}$ to q depends on the initial ratio $\frac{x_2}{x_1}$ and the parameters α and β.

If

$$\frac{x_2}{x_1} = q^1 = q^* ,$$

than there is a condition for all ratios of the series

$$\frac{x_k}{x_{k-1}} = q_{k-1} = q^* .$$

In case of geometric progression it's possible to find the solution with the help of two iterations.

Example 1. Let's consider the Fibonacci general series by means of the recurrent formula

$$x_k = 0,7 x_{k-2} + 0,4 x_{k-1}, \quad k \geq 3 .$$

The ratio $\frac{x_k}{x_{k-1}}$ with increasing k tends to greater root of the equation

$$q^2 - 0,4q - 0,7 = 0 .$$

The general series of Fibonacci for the initial values $x_1 = 1$, $x_2 = 1$ is in a range of cells A2:A18 (Table 2), and the ratio

$$q_\kappa = \frac{x_k}{x_{k-1}}$$

is in a range of cells B4:B18.

The deviation $l_k = q_k - q^*$ (a range of cells C4:C18) is alternating series of descending, moreover, it can also be considered as geometric progression, the denominator of which is equal to a given number -0,62.

Table 2. The Fibonacci general series, finding the limits of ratio and the second-order equation

	A	B	C	D	E	F	G
1	Fibonacci general series	=A2*D2+A3*E2		α	β		
2	1			0,7	0,4	=(E2^2+4*D2)^(1/2)	
3	1	**Ratio**	**Deviation**				
4	1,1	1,[=A4/A3]	0,03977	=B4-E5	1,72046505	$\beta + \sqrt{\beta^2 + 4\alpha}$	
5	1,14	1,03636	-0,02387		1,06023253		
6	1,226	1,07544	0,01521				
7	1,2884	1,05090	-0,00934		=(E2+E4)/2		
8	1,37356	1,06610	0,00586	-0,60021	=C5/C4		
9	1,45130	1,05660	-0,00363	-0,63707			
10	1,54201	1,06250	0,00227	-0,61392			
11	1,63272	1,05882	-0,00141	-0,62826			
12	1,73250	1,06111	0,00088	-0,61930			
13	1,83590	1,05969	-0,00055	-0,62486			
14	1,94711	1,06057	0,00034	-0,62139			
15	2,06397	1,06002	-0,00021	-0,62355			
16	2,18857	1,06036	0,00013	-0,62221			
17	2,32021	1,06015	-0,00008	-0,62305			
18	2,46008	1,06028	0,00005	-0,62252			

As $q_k - q^*$ and $q_{k-1} - q^*$ (Fig. 1) have different signs than $\left| q_k - q^* \right| \le \left| q_k - q_{k-1} \right|$ and sequence of ratios

$$q_\kappa = \frac{x_k}{x_{k-1}}$$

match to the root q^*.

Thus, the root of the equation can be found using the general series of Fibonacci.

The required accuracy of the root can be specified by the inequality

$$\left| q_k - q_{k-1} \right| \le \varepsilon$$

As,

$$\lim_{k \to \infty} \frac{l_k}{l_{k-1}} = -0,62$$

(the range of cells D8:D18) , than the remainder is a decreasing geometric progression with a negative denominator and the error does not exceed the last member of rejection.

The root values of the equation and boundary of ratio coinsides, so there is a condition

$$\lim_{k \to \infty} \frac{x_k}{x_{k-1}} = \frac{\beta + \sqrt{\beta^2 + 4\alpha}}{2}$$

Let's consider a general view of a Fibonacci series with the recurrence formula

$$x_k = a_1 x_{k-1} + a_2 x_{k-2} + a_3 x_{k-3}, \quad k \ge 4$$

Each succeeding element equals to the sum of the three previous elements of the sequence which are relatively multiplied by $a_1; a_2; a_3$.

It's supposed to believe there is a sequence executed proportion:

$$\frac{x_k}{x_{k-1}} = \frac{x_{k-1}}{x_{k-2}} = \frac{x_{k-2}}{x_{k-3}} = q,$$

where

$$x_{k-2} = q x_{k-3},$$

$$x_{k-1} = q x_{k-2} = q^2 x_{k-3}$$

From the recurrence formula we obtain

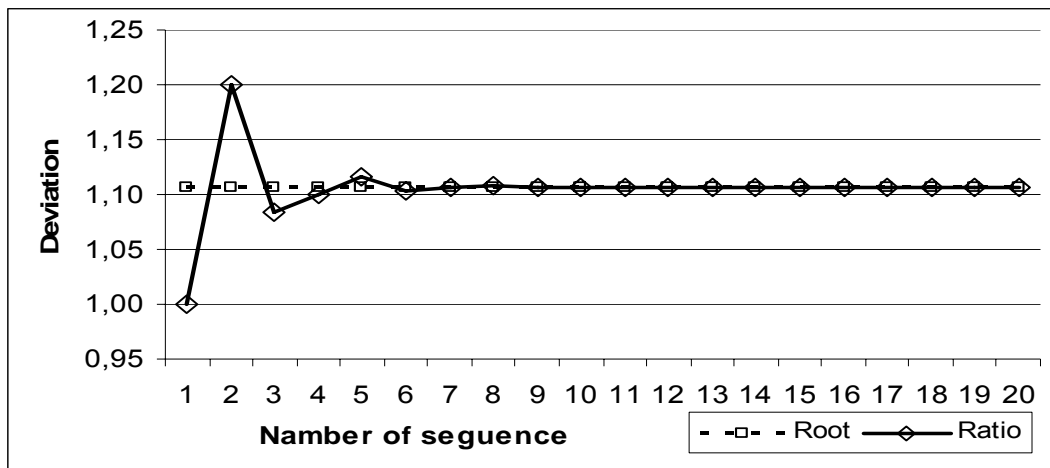$$\frac{x_k}{q^2 x_{k-3}} = \frac{a_1 q^2 + a_2 q + a_3}{q^2} = q,$$

Fig. 1. Elements deviations of the sequence q_k from q^*

from which we obtain a cubic equation

$$q^3 - a_1 q^2 - a_2 q - a_3 = 0 .$$

The solution of cubic equation can be found by Cardano`s formulas.

After substitution

$$q = y + \frac{a_3}{3},$$

we get

$$y^3 + py + c = 0 ,$$

$$p = -a_2 - \frac{a_3^2}{3};$$

$$c = -a_1 + \frac{2a_2^2}{27} - \frac{a_1 a_3}{3}.$$

Solution has the form of

$$y = \sqrt[3]{-\frac{c}{2} + \sqrt{\frac{c^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{c}{2} - \sqrt{\frac{c^2}{4} + \frac{p^3}{27}}} .$$

Returning to the previous variable we obtain

$$q = y + \frac{a_3}{3} .$$

Thus, the sequence

$$q_\kappa = \frac{x_k}{x_{k-1}},$$

where

$$x_k = a_1 x_{k-1} + a_2 x_{k-2} + a_3 x_{k-3}$$

identical to the value

$$q^* = \frac{a_3}{3} + \sqrt[3]{-\frac{c}{2} + \sqrt{\frac{c^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{c}{2} - \sqrt{\frac{c^2}{4} + \frac{p^3}{27}}} .$$

Example 2. Let's find the solution of the third order.

Consider the general Fibonacci series for initial values $x_1 = 1$, $x_2 = 1$, $x_3 = 1$ a recurrence formula is

$$x_k = 0,3 x_{k-1} + 0,4 x_{k-2} + 0,5 x_{k-3}, k \geq 4 .$$

General Fibonacci series are shown in the range of cells A2:A17 (Table 3) and the ratio

$$q_\kappa = \frac{x_k}{x_{k-1}}$$

in a range of cells B4:B17.

In cell A5 enter the recurrence formula

$$x_k = 0,3 x_{k-1} + 0,4 x_{k-2} + 0,5 x_{k-3}$$

(=A2*D2+A3*E2+A4*F2)

Then, using the above described iterative formula, we find series of values

$$x_k = a_1 x_{k-1} + a_2 x_{k-2} + a_3 x_{k-3} , k \geq 4$$

and simultaneously find a series of values

$$q_\kappa = \frac{x_k}{x_{k-1}} .$$

Repeat this operation as long as the condition is not fulfilled

$$\left| q_k - q_{k-1} \right| \leq \varepsilon ,$$

there is the difference between current and previous ratio which is considered as less than the specified error.

Table 3. Using General Fibonacci series in which each next element is the sum of the previous three, multiplied by the corresponding coefficients

	A	B	C	D	E	F	G
1	Fibonacci series	Ratio	Deviation	a1	a2	a3	
2	1	=A2*C2+A3*D2+A4*E2		0,3	0,4	0,5	
3	1	1		a1	a2	a3	
4	1	1		-0,3	-0,4	-0,5	
5	1,2	1,2					
6	1,300	1,08333	-0,02318				
7	1,430	1,10000	-0,00652	p=	-0,483333	=D4-E4^2/3	
8	1,595	1,11538	0,00887	c=	-0,375926	=C4+2*E4^3/27-E4*D4/3	
9	1,760	1,10313	-0,00338	q^2/4+p^3/27=	0,031148	0,176488	=D9^(1/2)
10	1,947	1,10642	-0,00009	=D8^2/4+D7^3/27	0,364451	0,714299	=D10^(1/3)
11	2,156	1,10732	0,00080		0,011475	0,225551	=D11^(1/3)
12	2,384	1,10610	-0,00042	=-D8/2+E9	=-D8/2-E9	y=0,939850	
13	2,638	1,10657	0,00005		q=1,106517	=E10+E11	
14	2,920	1,10658	0,00006	0,0000000	0,000000	=E12-E4/3	
15	3,231	1,10647	-0,00005	=B27^3+E4*B27^2+D4*B27+C4			
16	3,575	1,10653	0,00001				
17	3,955	1,10652	0,00000	=E13^3-E2*E13^2-D2*E13-C2			

Ratio $\dfrac{x_k}{x_{k-1}}$ tends to increase with larger root of the equation

$$q^3 - 0,3q^2 - 0,4q - 0,5 = 0 .$$

As $q_k - q^*$ and $q_{k-1} - q^*$ (Fig. 2) have different signs, than

$$\left| q_k - q^* \right| \le \left| q_k - q_{k-1} \right|$$

and the sequence of correlation

$$q_\kappa = \frac{x_k}{x_{k-1}}$$

identical to the root q^*.

Thus, the root of the equation can be found using the general Fibonacci series.

The required accuracy of the root can be specified by the following inequality

$$\left| q_k - q_{k-1} \right| \le \varepsilon .$$

Example 3. Let's find the solution of the fourth order equation.

To solve the equation, such as the equation of the fourth order

$$q^4 - a_1 q^3 - a_2 q^2 - a_3 q - a_4 = 0 ,$$

we should use the recurrence formula (general series of Fibonacci numbers)

$$x_k = a_1 x_{k-1} + a_2 x_{k-2} + a_3 x_{k-3} + a_4 x_{k-4} .$$

and find the boundary of ratio

$$q^* = \lim_{k\to\infty} \frac{x_k}{x_{k-1}} .$$

Example 4. Let's consider an example with the following coefficients:

$$a_1 = 0,5; \quad a_2 = 0,7; \quad a_3 = 0,5; \quad a_4 = 0,6$$

namely

$$q^4 - 0,5q^3 - 0,7q^2 - 0,5q - 0,6 = 0 .$$

We should enter the recurrence formula

$$x_5 = 0,5x_4 + 0,7x_3 + 0,5x_2 + 0,6x_1$$

(=A2*H3+A3*G3+A4*F3+A5*E3)

(Table 4) to the cell A6.

After you copy it down the column, get the general series of Fibonacci.

We should enter the formula of the finding the ratio element to previous element of this series to find the root of the equation in cell B6 $\dfrac{x_5}{x_4}$ (=A6/A5).

Than we copy the formulas to the columns until we get the required number of identical signs (required accuracy) in the second column (the ratio).

We, consider the following function

$$f_1(q) = \frac{f(q)}{q - q_1}$$
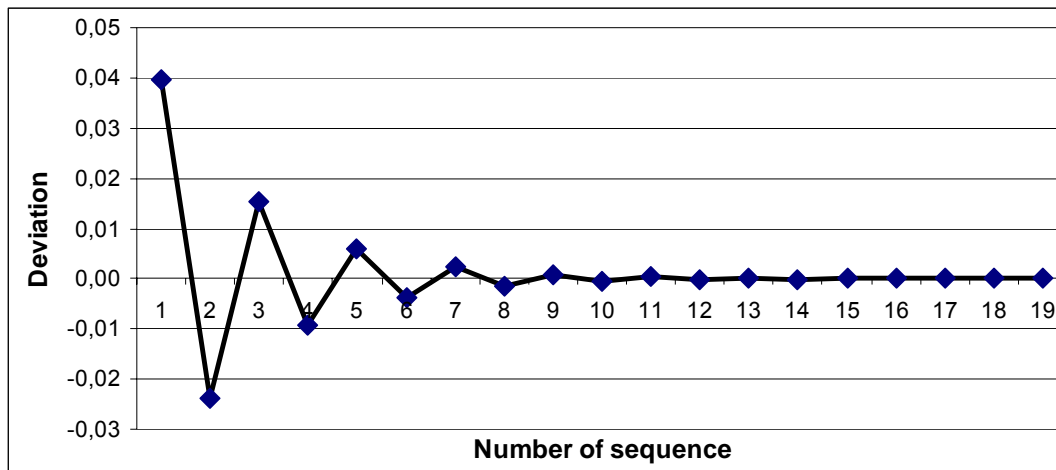
to find the root of the equation.

Fig. 2. Convergence of sequences of ratios

The received polynomial $f_1(q)$ has the same root as the polynomial $f(q)$, with the exception of the first root and will have order which is less on the unit.

Than we find the coefficients of the polynomial using the method of Horner in a range of cells (D5: H8).

Than we enter the coefficients of the polynomial

$$f(q) = q^4 - 0,5q^3 - 0,7q^2 - 0,5q - 0,6$$

for this range of cells (D5: H5) and enter

$$q_1 = 1,434364$$

to the cell C6.

The first coefficient of the polynomial $f_1(q)$ will be equal to the first coefficient $f(q)$, the other coefficients we find using the formula

$$b_{i+1} = q_1 b_i + a_{i+1} \quad (=\$C6*E6+F5),$$

where

$$f(q) = a_0 q^n + a_1 q^{n-1} + \cdots + a_n;$$

$$f_1(q) = b_0 q^{n-1} + b_1 q^{n-2} + \cdots + b_{n-1}.$$

We get the equation of the third order

$$q^3 + 0,9344q^2 + 0,6402q + 0,4183 = 0.$$

Table 4. The usage of the Fibonacci general series where each next element equals to the sum of the previous four elements which are multiplied by the corresponding

	A	B	C	D	E	F	G	H
1	**Series**	**Ratio**			**Coefficients**			
2	1	=A2*\$H\$3+A3*\$G\$3+A4*\$F\$3+A5*\$E\$3			a1	a2	a3	a4
3	2			**Verification**	0,5	0,7	0,5	0,6
4	3		1,434364	0,000000	**Homer's method**			
5	4	=C4^4-E3*C4^3-F3*C4^2-G3*C4-H3		1	-0,5	-0,7	-0,5	-0,6
6	6	1,4250	1,434364	1	0,9344	0,6402	0,4183	0,0000
7	8	1,4649	-0,792529	1	0,1418	0,5278	0,0000	0,0000
8	12	1,4329	=\$H\$9*D10+\$G\$9*D11+\$F\$9*D12		a1	a2	a3	
9	17	1,4273		**Series**	**Ratio**	-0,9344	-0,6402	-0,4183
10	25	1,4352		-1,64634E-09	=\$C6*E6+F5			
11	35	1,4362		1,30442E-09				
12	50	1,4339	=A12/A11	-1,03267E-09	-0,791668		D=	-2,09112
13	72	1,4340		8,18448E-10	-0,792554			
14	104	1,4345		-6,4924E-10	-0,793258	=E7^2-4*D7*F7		
15	149	1,4345		5,14612E-10	-0,792638			
16	214	1,4343		-4,07541E-10	-0,791937	=D16/D15		
17	306	1,4343		3,22907E-10	-0,792331			
18	440	1,4344		-2,56063E-10	-0,792992			

We enter the recurrence formula

$$x_4 = -0,9344x_3 - 0,6402x_2 - 0,4183x_1$$

(=A2*H3+A3*G3+A4*F3+A5*E3)

to the cell D13.

The square equation

$$q^2 + 0,1418q + 0,5278 = 0$$

has the negative discriminant.

Example 5. We consider the solving of the equation of the seventh order (Table 5):

$$q^7 - 0,5q^6 - 0,1q^5 + 0,4q^4 - 0,6q^3 + 0,3q^2 - -0,8q + 0,2 = 0$$

We enter the recurrence formula

$$x_k = 0,5x_{k-1} + 0,1x_{k-2} - 0,4x_{k-3} + 0,6x_{k-4} - -0,3x_{k-5} + 0,8x_{k-6} - 0,2x_{k-7}$$

(=A2*K3+A3*J3+A4*I3+A5*H3+A6* G3+A7*F3+A8*E3) to the cell A9.

Then we find the coefficients of the equation in the range of cells E3: K3.

According to the given algorithm we find the root of the equation.

It is necessary to note that finding of the equation root of higher order needs higher quantity of iterations.

4. Conclusions

1. It has been applied the concept of the general series of Fibonacci, the recurrence formula of the population changes upon condition at the factors saving.

2. It is shown that the recurrence formula corresponds to the equation, the largest root index

identities the increase (decrease) population, that shows how many times the population increases (decreases) in the current period compared to the previous ones.

3. The population rate of change with the number of periods is independent from the initial conditions (initial size of population), and depends on coefficients

4. The algorithm for the solving equation of the *n*-th order

$$q^n - a_1q^{n-1} - \cdots - a_{n-2}q^2 - a_{n-1}q - a_n = 0$$

with the method of the Fibonacci series consists of the original equation that should be adjusted to the form

$$q^n = a_1q^{n-1} + \cdots + a_{n-2}q^2 + a_{n-1}q + a_n,$$

we should apply the recurrence formula

$$x_k = a_1x_{k-1} + a_2x_{k-2} + \cdots + a_{n-1}x_{k-n+1} + a_nx_{k-n},$$

$$k \geq n+1,$$

than it is necessary to apply the ratio

$$q_\kappa = \frac{x_k}{x_{k-1}}.$$

Iterative elements of the series are searched by as long as the condition is not fulfilled

$$|q_k - q_{k-1}| \leq \varepsilon,$$

where ε – given allowable error.

References

[1] http// ukr.wikipedia.org/Newton's metod.

[2] http// ru.wikipedia.org/Fibonacci numbers.

[3] *Volkov, E. A.* Numerical methods. Moscow, Fizmatlit. 2003 (in Russian).

Table 5. The usage of the Fibonacci general series where each next element equals to the sum of the previous seven members which are multiplied by the corresponding coefficients

	A	B	C	D	E	F	G	H	I	J	K
1	**Series**	**Ratio**			**Coefficients**						
2	-7892				a1	a2	a3	a4	a5	a6	a7
3	8964				0,5	0,1	-0,4	0,6	-0,3	0,8	-0,2
4	-9803				**Homer's method**						
5	11102			1	-0,5	-0,1	0,4	-0,6	0,3	-0,8	0,2
6	-12173	-1,0965	-1,114	1	-1,6137	1,6971	-1,4899	1,0593	-0,8797	0,1796	0,0000
7	13752	-1,1297				=$C6*E6+F5					
8	-15114	-1,0990									
9	17039	-1,1273	=A2*K3+A3*J3+A4*I3+A5*H3+A6*G3+A7*F3+A8*E3								
10	-18762	-1,1012									

Є.Ю. Толбатов. Розв'язання рівнянь з використанням загального ряду Фібоначчі

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Роглянуто ефективний метод розв'язання рівняння з використанням загальних рядів Фібоначчі. Введено поняття загального ряду Фібоначчі. Наведено рекурентну формулу зміни обсягу популяції. На основі рекурентних формул отримано рівняння, найбільший корінь якого є індексом зміни обсягу популяції. В електронних таблицях Excel зроблено приклади розв'язання лінійних рівнянь із використанням загальних рядів Фібоначчі.

Ключові слова: границя відношення; збіжність ряду; індекс зміни; популяція; рівняння; ряд Фібоначчі.

Е.Ю.Толбатов. Решение уравнений с использованием общего ряда Фибоначчи

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Рассмотрен эффективный метод решения уравнения с использованием общих рядов Фибоначчи. Введено понятие общего ряда Фибоначчи. Приведена рекуррентная формула изменения объема популяции. На основе рекуррентных формул получено уравнение, наибольший корень которого является индексом изменения объема популяции. Примеры решения линейных уравнений сделаны в электронных таблицах Excel с использованием общих рядов Фибоначчи.

Ключевые слова: индекс изменения; популяция; предел отношения; ряд Фибоначчи; сходимость ряда; уравнения.

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