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METHOD OF SOLVING SYSTEMS OF LINEAR ALGEBRAIC EQUATIONS IN THE DISTRIBUTED CALCULATING ENVIRONMENT

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The method of numerical solving of systems of linear algebraic equations produced by the tasks of aerodynamics and hydrodynamics is offered in the paper. The method makes use of the auxiliary function, a value of which is the measure of misclosure of the achieved approximate solution. Proposed a modification of the method suitable for the use in the distributed computing environment.

Запропоновано метод числового розв'язання систем лінійних алгебричних рівнянь, породжених задачами аеро- та гідродинаміки, з використанням допоміжної функції, значення якої є мірою відхилу отриманого наближеного розв'язку. Розглянуто модифікацію методу, призначену для використання в розподіленому обчислювальному середовищі.

Предложен метод численного решения систем линейных алгебраических уравнений, порожденных задачами аэро- и гидродинамики, с использованием вспомогательной функции, значение которой является мерой невязки полученного приближенного решения. Предложена модификация метода, предназначенная для использования в распределенной вычислительной среде.

Statement of purpose

A number of problems of aircraft motion control and other practically important problems of aero- and hydrodynamics require to take into account the nonstationarity of aerodynamic characteristics of controlled objects. Applying simplified aerodynamic models leads in such cases to the inadequate results. On the other side, the use of complete models is difficult because of the large dimensionality of the systems of equations in these problems; thus there are a number of issues which do not allow utilizing possibilities of modern computer technique effectively. Among them are shortage of main memory of computer and considerable duration of calculations.

In the common case, the aforementioned problems are reduced to the problems of numerical solving of the systems of linear algebraic equations of large dimensions. There is vast literature, devoted to the methods of applied linear algebra, and software products which realize the most popular algorithms of calculative linear algebra. Though, due to the dimensions of contemporary problems, the volumes of the data, that it is necessary to keep and process at their solving, grow more rapidly than the amount of the operative memory of the available computers, and more rapidly than their speed characteristics.

An additional factor of increase of dimension of these problems is their substantial three-dimensionality. In the case of symmetric geometry of problems at not very large values of the Reynolds number, numerical modeling appears fully successful on the basis of the two-dimensional nonstationary Navier-Stokes equations. However, at large Reynolds numbers three-dimensional stochastic turbulent fluctuations are imposed substantially on the two-dimensional non-stationary flow, that is why numerical modeling must be conducted in the three-dimensional problem setting even for simple symmetric geometry of a flow. As it was shown in [1], the modeling of the detailed

structure of three-dimensional motion of viscose gas is possible only with application of modern multiprocessor calculative complexes or distributed calculative environments. Using the latter feature is the contemporary actual direction of development of calculative mathematics. Besides, distributed computer system exists on boards of many modern aircrafts. Consequently, their use will allow solving problems of aircraft control using its onboard calculating facilities.

Thus, there appears a problem of developments of new mathematical methods which would allow solving the systems of linear algebraic equations of large dimensions with the less charges of machine time, comparatively with existing methods, and it is desirable that the offered methods be suitable for implementation in the calculating cluster.

Analysis of researches and publications

The indicated contradictions motivate researchers to offer new approaches in such directions.

1. Creation of new models of processes of gas dynamics, in particular, with the purpose of simplification of necessary calculations [2; 3].

2. Creation of new methods of solving problems of linear algebra. In particular, the economic direct methods were created for the number of particular cases of the systems with sparse matrix [4], a number of iterative methods [5; 6]. Also proposed approaches for solving problems, in which the conduct of the systems is described or determined with distributed parameters, based on the theory of the first and second Lyapunov methods [7; 8].

3. Attempts to build the algorithm of calculation so that it could be divided into separate loosely connected parts, which can be implemented on separate computers of distributed computing network.

The given article is devoted to the questions which belong to the second and third of the listed directions. The unsolved problems of existent approaches are the following. The most of proposed methods of solving of such systems are developed in supposition that the matrix of the system has a band structure, and the width of the band is considerably less than the order of matrix. However this supposition is applicable not for all problems. In addition, the proposed methods allow increasing the order of the solved system only on condition of the proper increase of volume of the computer system resources which must be utilized for its solving.

The hydrodynamics problems in many cases are reduced to the systems of equations with sparse matrices. Exactly for such problems high efficiency of iterational methods can be attained as compared to direct methods, as during the process of iterations the matrix of the system does not change, that is it remains sparse. The use of direct methods in such cases is ineffective. For instance, application of method of solving with exclusion leads to the substitution of large number of zero elements with non-zero ones, so the matrix looses the property of sparsity.

The aim of the work

It is possible to make a conclusion, that actual is a search of new iterational methods of solving of the problems in question, oriented to work with the systems of equations of large dimensionalities. Using the Lyapunov's functions method seems to be a perspective direction for such a search.

It of is why the aim of this paper is to propose a method of solving the systems of linear algebraic equations of large dimensionalities, based on the method of the Lyapunov's functions, and to analyze the possibility of its application in distributed computing environment.

Iterational method of solving a system of linear algebraic equations

Let the system of n linear algebraic equations is solved

$$\mathbf{AX} = \mathbf{B}, \qquad (1)$$

where **A** is a given constant matrix is set with the real elements of the dimension $n \times n$;

B is a given constant vector of right parts of equations (a vector with the real elements of the dimension *n*);

X is a vector with real components (the vector of solutions) of the dimension *n*, the exact value of which must be found as a result of solving the problem.

For each vector **X** one can find from the system (1) a vector-function of the misclosure, let's designate it as **M**(**X**) :

$$\mathbf{M}(\mathbf{X}) = \mathbf{AX} - \mathbf{B}.$$

Let us introduce some constant, symmetrical weight matrix **Q**, $\mathbf{Q} = \mathbf{Q}^{\mathbf{T}}$.

Let's introduce a scalar auxiliary function V(**X**) as a quadratic norm of the misclosure vector, found with the weight matrix of **Q**:

$$\mathrm{V}(\mathbf{X}) = (\mathbf{AX}-\mathbf{B})^{\mathbf{T}}\mathbf{Q}(\mathbf{AX}-\mathbf{B}) = \mathbf{M}(\mathbf{X})^{\mathbf{T}}\mathbf{QM}(\mathbf{X}), \quad (2)$$

If the value of the vector **X** is an approximate solution of the system of equations (1), got on the certain step of the procedure, then the value of the function V(**X**) is the measure of misclosure of this approximate solution.

Let us introduce the two differential equations, which determine the desired conduct of the auxiliary function V(**X**) and the vector of solutions **X** in the process of solution:

$$\dot{\mathrm{V}} + c\mathrm{V} = 0, \quad (3)$$

where *c* is some constant value, which determines the speed of diminishing of the misclosure norm (it may be chosen in an arbitrary way);

$$\dot{\mathbf{X}} = -\mathrm{k}\frac{\partial \mathrm{V}}{\partial \mathbf{X}}, \quad (4)$$

where k = k(**X**) is a function of the state vector, expression for which it is possible to obtain from taking into account the requirement of compliancy of the equations (1)–(4):

$$\mathrm{k}(\mathbf{X}) = \frac{c(\mathbf{AX}-\mathbf{B})^{\mathbf{T}}\mathbf{Q}(\mathbf{AX}-\mathbf{B})}{4(\mathbf{AX}-\mathbf{B})^{\mathbf{T}}\mathbf{QAA}^{\mathbf{T}}\mathbf{Q}(\mathbf{AX}-\mathbf{B})}. \quad (5)$$

Using of the approach, proposed in [9], from the relations (2) – (5) taking into account the derivate of auxiliary function by virtue of **X** we'll get a system of differential equations for the components of the vector **X**, which in vector-matrix form may be presented as:

$$\dot{\mathbf{X}}_t = \mathbf{R}(\mathbf{X}_t), \quad (6)$$

where $\mathbf{X}_t$=**X**(*t*) is a vector-function of time, with the limit value at $t\to\infty$ equal to the vector of exact solutions of the system (1);

$$\mathbf{R}(\mathbf{X}) = -\mathrm{k}\frac{\partial \mathrm{V}}{\partial \mathbf{X}} =$$

$$= -\frac{c}{2}\cdot\frac{\mathbf{M}^{\mathbf{T}}(\mathbf{X})\mathbf{QM}(\mathbf{X})}{\mathbf{M}^{\mathbf{T}}(\mathbf{X})\mathbf{QAA}^{\mathbf{T}}\mathbf{QM}(\mathbf{X})}\mathbf{A}^{\mathbf{T}}\mathbf{QM}(\mathbf{X}) \quad (7)$$

is a vector-function, the vector of the right parts of the system of equations (6).

The procedure of obtaining the solution of the system (1) is reduced to integration of equation (6), as a result of which we find the vector-function which coincides to exact solution of the system of equations (1).

Dividing into independent subproblems for implementation of parallel calculations

Let us consider the simplest method of search of the solution of (6) – the numerical integration of the equation (6) by the Euler's method with the constant step:

$$t_{(i)} = t_{(0)} + i\cdot\Delta t \; ; \; \mathbf{X}(t_{(i)}) = \mathbf{X}_{(i)};$$

$$\mathbf{X}_{(i+1)} = \mathbf{X}_{(i)} + \mathbf{R}(\mathbf{X}_{(i)})\cdot\Delta t, \quad (8)$$

where lower indexes in brackets mark the number of the step to which that value corresponds.

For such an algorithm every step of iteration includes a calculation on the formula (8), which requires the following sequence of actions:

1) calculation of the vector-function **R**(**X**) on the formula (7), which includes a set of operations of matrix arithmetic, which are executed with matrices and vectors of the dimension *n* (*n* is the dimension of the system (1));

2) calculation of vector of the product $\mathbf{R}(\mathbf{X}_{(i)})\cdot\Delta$t of the dimension *n*;

3) calculation of the sum of the vectors $\mathbf{X}_{(i)}$ and $\mathbf{R}(\mathbf{X}_{(i)})\cdot\Delta$t of the dimension *n*.

The calculation of numerator and denominator of expression (7) requires multiplying the matrices, therefore at the direct algorithm of multiplication the amount of arithmetic operations will be of the order of n^2.

With the purpose of breaking the procedure of calculation into subproblems the following approach is offered. Let's break up the matrices and vectors of the problem into the blocks:

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \dots & \mathbf{A}_{1p} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \dots & \mathbf{A}_{2p} \\ \dots & \dots & \dots & \dots \\ \mathbf{A}_{p1} & \mathbf{A}_{p2} & \dots & \mathbf{A}_{pp} \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \dots \\ \mathbf{B}_p \end{pmatrix}$$

where $\mathbf{A}_{11}$, $\mathbf{A}_{22}$, … $\mathbf{A}_{pp}$ are square submatrices of the matrix $\mathbf{A}$, of the dimensions $r_1, r_2, \dots, r_p$;

$\mathbf{B}_1$, $\mathbf{B}_2$, … $\mathbf{B}_p$ are subvectors, component parts of the vector $\mathbf{B}$, which have dimensions equal to those of the blocks of the matrix $\mathbf{A}$.

Let us break the matrices $\mathbf{A}^{\mathbf{T}}$ and $\mathbf{Q}$, vectors $\mathbf{X}$, $\mathbf{N}$ and others, obtained in the course of calculations, in the same manner.

Now let's divide the iterational process into a number of steps on the basis of these blocks. At the first step let's consider the problem:

$$\mathbf{A}_{11}\mathbf{X}_1 = (\mathbf{B}_1 - \mathbf{A}_{12}\mathbf{X}_2 - \mathbf{A}_{13}\mathbf{X}_3 - \dots - \mathbf{A}_{1p}\mathbf{X}_p). \quad (9)$$

Let us assume that the sought vector now is $\mathbf{X}_1$, and the other subvectors of the vector X (the vectors $\mathbf{X}_2$, $\mathbf{X}_3$, … , $\mathbf{X}_p$) are assumed constant and equal to their approximate values, found by now; in particular, on the first step of algorithm – to the initial approximations $\mathbf{X}_{2\,(0)}$, $\mathbf{X}_{3\,(0)}$, … , $\mathbf{X}_{p\,(0)}$. Then equation (9) acquires the sense analogical to sense of the equation (1), it is the system of linear algebraic equations of the dimension r_1 with the vector of unknown values $\mathbf{X}_1$. This problem may be solved using the method proposed above, that is, we write for it the equation analogous to (6), and integrate it by the Euler's method. It is enough to make one step in the numerical integration, as a result we get for the $\mathbf{X}_1$ its first approximation $\mathbf{X}_{1\,(1)}$.

Now let's pass to the second step of iterational process. Now solve the system of equations

$$\mathbf{A}_{22}\mathbf{X}_2 = (\mathbf{B}_2 - \mathbf{A}_{11}\mathbf{X}_1 - \mathbf{A}_{13}\mathbf{X}_3 - \dots - \mathbf{A}_{1p}\mathbf{X}_p),$$

while consider $\mathbf{X}_2$ to be the sought vector, and the vectors $\mathbf{X}_1$ and $\mathbf{X}_3$, … , $\mathbf{X}_p$ assume to be known; as the value of $\mathbf{X}_1$ take the approximation $\mathbf{X}_{1\,(1)}$, found on the previous step of the algorithm, and for the values of vectors of $\mathbf{X}_3$, … , $\mathbf{X}_p$ take the approximations $\mathbf{X}_{3\,(0)}$, … , $\mathbf{X}_{p\,(0)}$. Executing one step like the previous one, find the approximation $\mathbf{X}_{2\,(1)}$.

Next, let's execute analogical steps for other parts of problems. After the complete cycle which consists of p steps the new approximations will be found for all the subvectors of the vector $\mathbf{X}$. Now it is possible again to pass to the search of the new approximation for $\mathbf{X}_1$, et cetera.

Conclusion

In the paper the method of finding of the solution of the system of linear algebraic equations (1) is proposed, as a limit, to which approaches the solution of the differential equation (6).

The method of dividing the process of calculations after the proposed method into poorly connected parts is offered, that allows implementing the proposed method in a calculative cluster.

During realization and subsequent development of the offered algorithm the following aspects are substantial and need additional researches.

1. The dimensionality of each of subproblems is less than the dimension of the initial problem. If the initial problem is broken into p identical blocks, the volume of calculations for implementation of one step during solving each of subproblems diminishes quadratically (as $O(p^2)$) comparatively with the volume of calculations for implementation of one step during solving of the full problem. At the same time the number of subproblems is a linearly growing value (it is equal to p). therefore after dividing into subproblems the general volume of calculations for passing one cycle diminishes linearly. However the general purpose of calculations is approaching of the

vector **X** to the exact solution, that is why it is necessary yet to compare the efficiency (by the speed of approaching) of one step of solving the complete problem and one cycle of solving the block-divided problem.

2. Each of the subproblems operates with constant matrices which are the blocks of the matrices **A**, **B**, **Q**. The communicational intensivity of the algorithm at implementation of calculations in a cluster is defined by the fact that only new approximations of the parts of the vector **X** are necessary data to pass from a subproblem to a subproblem at every step of the calculation. It is necessary to determine the volumes of transferred information for the different topologies of network and modes of transmission (for example, use of the multicast mode).

3. On the basis of the basic indexes, accepted in the theory of parallel algorithms, and taking into account a specific computer network structure, it is necessary to develop the complex index of calculative cost of this algorithm, which must also take into account the advantages of the customer – the user of the system, and define whether there is an optimal size for the pieces of problems according to this index.

4. Because the presence of set of blocks allows to choose the various methods of organization of cycles of iterational process, it is necessary to find the optimal among these methods.

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