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SPECIFIC SOLUTIONS OF ODE SYSTEM FOR MODELING STATIONARY COMPRESSIBLE FRICTION FLOW OF PERFECT GAS IN CONSTANT-AREA CHANNEL

In the paper some specific solutions of the general solution taken in [1] are considered to model a one-dimensional stationary flow of a compressible gas in the constant-area channel. New analytical solutions and interpretations of the known ones are given.

In [1] the general solution of the following autonomous ODE (ordinary differential equation system) is demonstrated which has been obtained also in [1]. It describes a stationary flow of perfect gas with the given constant heat flux, friction and mass forces that can perform work in the constant-area channel:

$$\frac{dP}{dx}=\frac{-\frac{1}{P}w_x-\frac{2\lambda}{\omega D_\text{h}}\frac{T}{P}\left[c_p+\left(\frac{gRG}{F}\right)^2\frac{T}{P^2}\right]+\left(\frac{F}{gRG}\right)^2 f_x\frac{P}{T}\left[c_p+\left(\frac{gRG}{F}\right)^2\frac{T}{P^2}\right]}{\left[c_p\frac{F^2}{gRG^2}+\left(gR-c_p\right)\frac{T}{P^2}\right]}; \tag{1}$$

$$\frac{dT}{dx}=\frac{\left[\frac{F^2}{gRG^2}-\frac{T}{P^2}\right]w_x-\frac{2\lambda}{\omega D_\text{h}}\left(\frac{gRG}{F}\right)^2\frac{T^3}{P^4}+f_x\frac{T}{P^2}}{\left[c_p\frac{F^2}{gRG^2}+\left(gR-c_p\right)\frac{T}{P^2}\right]}, \tag{2}$$

where P – pressure; λ – friction factor; $\omega=4$; T – temperature; c_p – specific heat at constant pressure; g – acceleration due to gravity; R – gas constant; f_x – mass force; G – flow-rate; D_h – hydraulic diameter; ρ – density; v – gas velocity; q_x – heat flux density referred to the mass flow-rate unity:

$$q_x=\frac{Q}{LG},$$

Q – heat flux; L – channel length. It is also shown that the energy ODE for such a general kind of flow takes the following form

$$\frac{dP}{dz}+R(z)P=Q(z), \tag{3}$$

which general solution is defined by the following integral [1]

$$P=e^{-\int R(z)dz}\left[\int Q(z)e^{\int R(z)dz}dz+\text{const}\right]; \tag{4}$$

$$R(z)=-\frac{c_p\frac{2\lambda}{\omega D_\text{h}}z-c_p\left(\frac{F}{gRG}\right)^2 f_x\frac{1}{z}}{\left\{\left[c_p\left(\frac{F}{gRG}\right)^2 f_x-\frac{F^2}{gRG^2}w_x\right]-c_p\frac{2\lambda}{\omega D_\text{h}}z^2\right\}};$$

$$Q(z)=\frac{w_x+\dfrac{2\lambda}{\omega D_h}\left(\dfrac{gRG}{F}\right)^2 z^2-f_x}{\left\{\left[c_p\left(\dfrac{F}{gRG}\right)^2 f_x-\dfrac{F^2}{gRG^2}w_x\right]-c_p\dfrac{2\lambda}{\omega D_h}z^2\right\}};$$

$$z=\frac{T}{P}.$$

Specific solutions of the ODE system (1)–(2) are identified by equating to zero the variables that contribute to the system different forces or energy forms, or its singularities. The denominator of the system (1)—(2) makes it singular under the following condition:

$$c_p+\left(1-\frac{c_P}{gR}\right)\left(\frac{gRG}{F}\right)^2\frac{T}{P^2}=0,$$

thence

$$G=\sqrt{\frac{c_P}{(c_P-gR)gRT}}PF. \quad (5)$$

Formula (5) is nothing but a kind of the second Newton law. Its square root addend has inverse velocity dimension, and PF corresponds to the force which pushes gas. Such a flow-rate value at which singularity takes place is called as critical with the gas velocity equal to sonic. This statement results from the critical flow-rate formula (5). In fact, let's rewrite it as follows

$$\left(\frac{P}{gRT}vF\right)^2=\frac{c_P}{c_v gRT}P^2F^2,$$

hence it follows from the above formula

$$v^2=\frac{c_P}{c_v}gRT,$$

i.e. is a definition of the local sonic velocity in gas flow. It should be noted that the critical flow-rate as a phenomenon has the same meaning and form for all flow models (with heat exchange or without, adiabatic or isentropic).

Let's substitute the critical flow-rate formula (5) in (3). Making some simplifications we obtain

$$\frac{dT}{dP}=-\frac{\dfrac{gR}{c_p}\dfrac{T}{P}w_x-\dfrac{2\lambda}{\omega D_h}gR\dfrac{c_p}{(c_p-gR)}\dfrac{T^2}{P}+f_x\dfrac{T}{P}}{w_x-f_x\dfrac{c_p}{gR}+c_p\dfrac{c_p}{(c_p-gR)}\dfrac{2\lambda}{\omega D_h}T}.$$

The last ODE one may easily convert to the form

$$\frac{dT}{dP}=\frac{gRT}{c_p P}.$$

Integration of the last ODE being one with separable variables gives

$$\int\frac{dT}{T}=\frac{gR}{c_p}\int\frac{dP}{P}+C,$$

then after some simple conversions we can obtain

$$\ln T=\frac{gR}{c_p}\ln P+C,$$

and finally we have

$$\frac{T}{P^{\frac{c_p}{gR}}} = \text{const}.$$

Thus pressure and temperature at a choke for the general case of a nonisentropic flow with heat exchange, mass forces influence and mass forces work are connected by the Poisson law.

The possibility to obtain the solution of the energy equation for the general flow case in the final form, i.e. in elementary functions, is achieved by solving two integrals In (4). The first of these integrals has the following form

$$\int R(z)dz = -\int \frac{c_p \frac{2\lambda}{\omega D_h} z - c_p \left(\frac{F}{gRG}\right)^2 f_x \frac{1}{z}}{\left\{\left[c_p \left(\frac{F}{gRG}\right)^2 f_x - \frac{F^2}{gRG^2} w_x\right] - c_p \frac{2\lambda}{\omega D_h} z^2\right\}} dz.$$

It is easily shown, that the given integral splits into two integrals

$$I_1 = \int \frac{z}{z^2 + a^2} dz, \tag{6}$$

$$I_2 = \frac{f_x}{\left(\frac{gRG}{F}\right)^2 \frac{2\lambda}{\omega D_h}} \int \frac{\frac{1}{z}}{z^2 + a^2} dz, \tag{7}$$

where

$$a^2 = \frac{\left(w_x - \frac{c_p}{gR} f_x\right)}{\frac{c_p}{gR}\left(\frac{gRG}{F}\right)^2 \frac{2\lambda}{\omega D_h}}. \tag{8}$$

Integrating (6) – (7) we have

$$\int R(z)dz = \ln z^{2b} \left|z^2 + a^2\right|^{\frac{1-2b}{2}},$$

where

$$b = -\frac{1}{2} \frac{\frac{c_p}{gR} f_x}{\left(w_x - \frac{c_p}{gR} f_x\right)}. \tag{9}$$

We consider the solution of the next integral in equation (4) that, when substituted in it the solution of the first integral obtained above, takes the following form

$$\int Q(z) \cdot e^{\int R(z)dz} dz = \int \frac{(w_x - f_x) + \frac{2\lambda}{\omega D_h}\left(\frac{gRG}{F}\right)^2 z^2}{\left[c_p \left(\frac{F}{gRG}\right)^2 f_x - \frac{F^2}{gRG^2} w_x\right] - c_p \frac{2\lambda}{\omega D_h} z^2} \cdot z^{2b} \left|z^2 + a^2\right|^{\frac{1-2b}{2}} dz.$$

After some simple transformations the last integral may be represented by a sum of the following integrals

$$I_3 = -\frac{\left(\frac{gRG}{F}\right)^2}{c_p} \int z^{2(b+1)} \left(z^2 + a^2\right)^{-\frac{1+2b}{2}} dz; \tag{10}$$

$$I_4 = -\frac{(w_x - f_x)}{c_p \dfrac{2\lambda}{\omega D_h}} \int z^{2b} \cdot \left(z^2 + a^2\right)^{-\frac{1+2b}{2}} dz \cdot \qquad (11)$$

The above integrals have binomial differentials as integral functions

Further, we obtain a general form of integrals (10, 11) under conditions $z > 0$, that is sufficient to obtain all the specific solutions as the above condition holds true. We convert integral I_3 with the following substitution

$$a^2 + z^2 = z^2 x^2; \quad z = a\sqrt{\frac{1}{\left(x^2 - 1\right)}}; \quad dz = -a\frac{x\sqrt{x^2 - 1}}{\left(x^2 - 1\right)^2}, \qquad (12)$$

then

$$I_3 = -\frac{\left(\dfrac{gRG}{F}\right)^2}{c_p} \int (-1)\left[\frac{a^2}{x^2 - 1}\right]^{(b+1)} \left[\frac{1}{\dfrac{a^2}{x^2 - 1} + a^2}\right]^{\frac{1+2b}{2}} a\frac{x\sqrt{x^2 - 1}}{\left(x^2 - 1\right)^2} dx \cdot$$

Simple transformations reduce the last integral to the form

$$I_3 = \frac{\left(\dfrac{gRG}{F}\right)^2}{c_p} a^2 \int \frac{1}{x^{2b}\left(x^2 - 1\right)^2} dx \cdot$$

Similar substitution into I_4 reduces to the following integral

$$I_4 = \frac{(w_x - f_x)}{c_p \dfrac{2\lambda}{\omega D_h}} a \int \frac{1}{x^{2b}\left(x^2 - 1\right)} dx \cdot$$

As one can see, both integrals have singularity at $x = 1$. It may occur at $a^2 = 1$, that is evident from substitution formula (12) used above, i.e. at

$$w_x - \frac{c_p}{gR} f_x = 0.$$

If gravitation forces are used as mass forces, this condition by considering these forces work, will have the form

$$q_x + \left(\frac{c_p}{gR} - 1\right) g \sin\alpha = 0,$$

in case this work is absent, we have

$$q_x + \frac{c_p}{gR} g \sin\alpha = 0. \qquad (13)$$

Hence we can make a conclusion that, the solution of the initial problem statement when mass forces and parameters invoking heat exchange are equal to zero each or compensate each other in a way the condition (13) holds true, leads to degeneration of the initial problem flow into nonisentropic adiabatic one.

Nonisentropic adiabatic flow in a constant section area channel. In this case from (6)–(11) follows

$$I_1 = \ln z; \quad I_2 = 0; \quad I_3 = -\frac{1}{2c_p}\left(\frac{gRG}{F}\right)^2 z^2; \quad I_4 = 0,$$

hence from (4) we have

$$P=\frac{1}{z}\left[-\frac{1}{2c_p}\left(\frac{gRG}{F}\right)^2 z^2+\text{const}\right] \tag{14}$$

being the equation which defines stagnation temperature $T^*=\text{const}$ of nonisentropic adiabatic flow.

The general solution of the intitial problem, obtained in [], when $w_x=f_x=0$ substituted, is converted in the following ODE

$$\frac{\frac{gR}{\lambda}\left(\frac{F}{gRG}\right)^2}{2D_h}\frac{\left\{c_p\text{const}^2+\text{const}\left(\frac{gRG}{F}\right)^2\left(1-\frac{c_p}{gR}\right)z^2+\frac{1}{2c_p}\left(\frac{gRG}{F}\right)^4\left(\frac{1}{2}-\frac{c_p}{gR}\right)z^4\right\}}{z^3\left[c_p\text{const}+\frac{1}{2}\left(\frac{gRG}{F}\right)^2 z^2\right]}dz=dx.$$

Integration of the last ODE gives

$$\left.\left\{-\frac{T^*}{(T^*-T)}+\left(1-\frac{2c_p}{gR}\right)\ln\left[\frac{(T^*-T)}{\frac{1}{2c_p}\left(\frac{gRG}{F}\right)^2}\right]\right\}\right|_1^2=2\frac{2c_p}{gR}\frac{2\lambda}{\omega D_h}L+\text{const}.$$

Taking into account that

$$c_P-c_V=gR;\quad k=\frac{c_P}{c_V};\quad 1-\frac{1}{k}=\frac{gR}{c_P};\ \omega=4,$$

finally, we have

$$\left.\left[\frac{k-1}{k+1}\frac{T^*}{(T^*-T)}+ln(T^*-T)\right]\right|_1^2=-\frac{2k}{k+1}\frac{\lambda}{D_h}L+\text{const}. \tag{15}$$

The solution (15) obtained in pressure-temperature variables is analogous to equation [2]

$$\frac{1}{\lambda_1^2}-\frac{1}{\lambda_2^2}-\ln\left(\frac{\lambda_2^2}{\lambda_1^2}\right)=\frac{2k}{k+1}\xi\frac{x}{D_h},$$

if we substitute

$$\lambda=\frac{v}{a_{\text{KP}}},\ a_{\text{KP}}=\sqrt{\frac{2k}{k+1}gRT^*},\ a^2=kgRT,\ \frac{T^*-T}{T}=\frac{v^2}{a^2}\frac{k-1}{2},$$

in (15).

Let's consider one more type of flow which generalizes nonisentropic adiabatic flow. For completeness sake this flow is supplemented with mass forces influence. At present, the analysis of such solution is absent in literature.

Nonisentropic adiabatic flow in a constant section area channel with mass forces. The following assumption corresponds to this case $w_x=0$. Then it is obvious

$\int R(z)dz=\ln z$.

Integral

$$\int Q(z)e^{\int R(z)dz}dz=-\frac{\left(\frac{gRG}{F}\right)^2}{c_p}\int\frac{z^3}{z^2+a_1^2}dz+\frac{f_x}{c_p\frac{2\lambda}{\omega D_h}}\int\frac{z}{z^2+a_1^2}dz,$$

is obtained at

$$b=\frac{1}{2}\quad\text{and}\quad a_1^2=-\frac{\dfrac{c_p}{gR}f_x}{\dfrac{c_p}{gR}\left(\dfrac{gRG}{F}\right)^2\dfrac{2\lambda}{\omega D_\text{h}}}.$$

Integration of the last integral and performing some reductions let us obtain the solution of the energy equation for this flow type. The solution is as follows

$$P=\frac{1}{z}\left\{-\frac{\left(\dfrac{gRG}{F}\right)^2}{2c_p}z^2+\left[\frac{\left(\dfrac{gRG}{F}\right)^2}{2c_p}a_1^2+\frac{1}{2}\frac{f_x}{c_p\dfrac{2\lambda}{\omega D_\text{h}}}\right]\ln\left|z^2+a_1^2\right|+\text{const}\right\}.$$

If we substitute in this solution the formula of a_1^2 (8), then the expression in brackets of it will be identically equal to zero, i.e. the stagnation formula for this flow type is analogous to the formula of nonisentropic adiabatic flow in case of mass forces absence. The result obtained is obvious immediately from the energy equation for this flow type. But such a procedure of getting solution let us test the validity of equations derived.

At $w_x=0$ and excluding pressure from (2) with the formula defining the stagnation temperature (14) and performing simple reductions we have

$$\frac{dT}{dx}=\frac{-\dfrac{2\lambda}{\omega D_\text{h}}4c_p\left(T^*-T\right)^2+2f_x\left(T^*-T\right)}{\left[gRT+2\left(gR-c_p\right)\left(T^*-T\right)\right]}.$$

The general solution of the above ODE is as follows

$$2\left(gR-c_p\right)T^*\int_1^2\frac{dT}{X}+\left[gRT+2\left(gR-c_p\right)\right]\int_1^2\frac{TdT}{X}=\int_1^2dx+\text{const},\tag{16}$$

where

$$X=-4c_p\frac{2\lambda}{\omega D_\text{h}}T^2+2\left(4c_p\frac{2\lambda}{\omega D_\text{h}}T^*-f_x\right)T+\left(2f_xT^*-4c_p\frac{2\lambda}{\omega D_\text{h}}T^*\right).$$

Integral solutions of (16) depend on the discriminant sign of the quadratic equation $X=0$. Let's verify the following inequality fulfillment

$$D=\left(4c_p\frac{2\lambda}{\omega D_\text{h}}T^*-f_x\right)^2+16c_p\frac{\lambda}{\omega D_\text{h}}T^*\left(f_x-2c_p\frac{2\lambda}{\omega D_\text{h}}\right)>0.$$

Taking the square and making some reductions, we have

$$\left(4c_p\frac{2\lambda}{D_\text{h}}\right)^2\left(T^*\right)^2+f_x^2-\left(4c_p\frac{2\lambda}{D_\text{h}}\right)^2T^*>0.$$

It is evident that the discriminant will always be positive, consequently, an analytical solution for nonisentropic adiabatic flow with mass forces influence will have the following form

$$\left[2\left(gR-c_p\right)T^*-\frac{B}{2}\left[gRT+2\left(gR-c_p\right)\right]\right]\frac{1}{\sqrt{D}}\ln\left|\frac{2A\cdot T+B-\sqrt{D}}{2A\cdot T+B+\sqrt{D}}\right|+\frac{1}{2A}\left[gRT+2\left(gR-c_p\right)\right]\times$$
$$\times\ln\left|AT^2+BT+C\right|=x+\text{const},$$

where

$$A=-4c_p\frac{2\lambda}{\omega D_\text{h}};$$

$$B = 2\left(4c_p \frac{2\lambda}{\omega D_h} T^* - f_x\right);$$

$$C = \left(2 f_x T^* - 4c_p \frac{2\lambda}{\omega D_h} T^*\right).$$

The solution obtained together with the equation defining the stagnation temperature forms a nonlinear algebraic equation system to determine pressures and temperatures in nonisentropic adiabatic flow with the given constant mass forces influence.

Thus, specific solutions to model stationary compressible gas flow in the constant section area channel are given in the paper, in particular, a new analytical solution without heat exchange but with a mass force. The specific solution of the energy equation of the initial problem with heat exchange is considered in the paper [3].

References

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РОЗРАХУНОК ВИМІРЮВАЛЬНОГО БЛОКА ВНУТРІШНЬОМОДЕЛЬНИХ ТЕНЗОМЕТРИЧНИХ ВАГ У СЕРЕДОВИЩІ MATHCAD

Запропоновано методику і алгоритм оптимізованого розрахунку на міцність і чутливість блока для виміру сили лобового опору та методику і алгоритм перевірних розрахунків вимірювального блока готових тензометричних ваг за заданими геометричними розмірами і навантаженням у середовищі MathCAD.

Вступ. Актуальною проблемою при створенні інформаційних технологій проектних досліджень (ІТПД) складних технічних об'єктів (ТО), зокрема, авіаційної техніки, є проектування первинних джерел інформації (ПДІ) [1; 2; 3]. До ПДІ відносять різні пристрої для визначення значень фізичних величин, зокрема, аеродинамічні тензометричні ваги (ТВ), які використовують для вимірювання значень складових моменту і сили при експериментальних дослідженнях (ЕД) моделей ТО, зокрема, моделі літального апарату (МЛА) в заданому діапазоні параметрів досліджень [4–6].

Найбільш поширеною схемою блока для виміру сили лобового опору є багатоланкова статично невизначена рама з двома вимірювальними балками [4; 6–8].

Проектування ТВ починається з вибору типу пружного блоку і розрахунку його параметрів, які значною мірою визначають характеристики ТВ. Параметри елементів пружного блока знаходять методом послідовних наближень міцності і чутливості. Але такий підхід не завжди призводить до бажаного результату. Тому доцільно застосовувати метод безпосереднього визначення характеристик ТВ, який дозволяє знайти параметри міцності і чутливості одночасно. У кінцевому підсумку проектуються ТВ з оптимальними параметрами, а також значно корочується час їх проектування.