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CLEAR AND FUZZY FRACTAL MODELS OF SPREADING DANGEROUS ENVIRONMENTAL PHENOMENA

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This article is devoted to investigation of possibility of widening models of spreading dangerous environmental phenomena, in particular Grassberger's models, on the base of notion of fuzzy fractal sets introduced by one of the authors. Basic concepts from the theory of fuzzy fractals are considered.

Розглянуто можливість розширення моделей розповсюдження небезпечних екологічних явищ, зокрема, моделей Грассбергера, на базі поняття нечіткої фрактальної множини. Наведено базові поняття з теорії нечітких фракталів.

Introduction

Today fractal models take a worthy place in ecological investigations. Moreover the fuzzy approach enriches and expands opportunities of realization of such models. This article is devoted to the problem of expansion of known fractal ecological models in clear terms on fuzzy space. The main objectives of this work is to investigate those problems of the theory of dangerous environmental processes which are natural and correct for application of fuzzy fractals.

Clear and fuzzy fractal models can be used for study of spreading of phenomena with small activity ranges at the beginning stage in ecological environments with elements of low mobility or static. Such models represent in some system expansion and attenuation of forest fires, lake pollution, initial stage of epidemic spreading etc. Similar approaches were studied by Malthus, Verhulst, Iden, Grassberger [1]. Schemes of such models include fractal clusters, which surfaces reflect nontrivial similarity laws. They can be studied from the point of view of fuzzy fractal theory [2] by describing clear-valued and fuzzy-valued fractal characteristics of the models.

Models of spreading dangerous phenomena with the use of clear fractals

In the clear sense of a model let us consider phenomena development according to the diagram

$$e_p + e_n \xrightarrow{z} e_p + e_p, \quad e_p \xrightarrow{n} e_n$$

with spreading coefficient z and neutralization coefficient n, where e_p is an element of environment with certain property, e_n is an element which does not have this property.

To widen model's applications it is possible to consider $\tilde{e}_p$ as an element with a certain property with membership function $\mu_p \to 1$, and $\tilde{e}_n$ – with membership function $\mu_n \to 1$.

At the same time z and n can also have fuzzy character, i.e. to be fuzzy numbers. If the ratio $\frac{z}{n}$ exceeds some critical value then "epidemic" situation occurs, that is, a stationary state with null density of elements e_p. Critical behavior of the system in some threshold **δ**-neighborhood can be described by directional percolation in (d + 1) – dimensional space, where d is the dimensionality of the space of the system under consideration.

There is another clear model:

$$e_p + e_c \xrightarrow{z} e_p + e_p, \qquad e_p \xrightarrow{b} e_s$$

with spreading coefficient z, where e_c is an element inclined to phenomenon perception, e_s is an element which left the system (did not perceive the phenomenon or perished as a result of the phenomenon). It is clear that it is more natural to consider "inclination" in a fuzzy sense, so, these elements in the fuzzy model also have respective values of membership function. Leaving the system by elements is characterized by the coefficient b. In a finite system the number of elements perceiving the phenomenon is not restored, so any phenomenon damps. But in the infinite case the phenomenon can spread for a long time. If the system elements are static a stationary percolation cluster appears, which has, depending on the model, a clear or fuzzy character.

It is possible to associate with a clear model a binary tree in which every element of the system occupies only one knot and the phenomenon spreads only by one step in time. A fuzzy model has a more complicated structure depending on the values of membership functions.

The time of initial phenomenon accumulation by the element can be neglected (the incubatory or latent period). In the clear model one element can be assumed to transfer the phenomenon to two nearest neighbors independently with probability p, it becomes obvious that it generates percolation cluster with directed connections. On the other hand, if the phenomenon is transferred to both nearest neighbors, or not transferred to any one, knot percolation takes place. Critical parameters in such a model are calculated in different ways, for example, with the similarity method with a final scale, expansion in series, the method of Monte Carlo.

For $p \approx p_c$ fractal dimension of the cluster which perceived the phenomenon is estimated by the formula :

$$d_F = d - \frac{\beta}{\delta} + \tau,$$

where τ is infinitively small value.

It is calculated that $d_F = 0{,}749 \pm 0{,}001$ for $d = 1$ and $d_F = 1{,}119 \pm 0{,}004$ for $d = 2$. It is considered that $d_F = \frac{3}{4}$ in one-dimensional case, but the exact value of fractal dimensionality is not known despite the model simplicity. Therefore application of fuzzy fractals for estimation of the mentioned parameters is natural.

The basic concepts of fuzzy fractals theory

The article is based on the works of scientists, among whom is one of the authors of this article, actively developing the theory of fuzzy fractals [3; 4; 5; 6].

The fuzzy set $\widetilde{A}$ on the universal set $U = \{u\}$ is the set of pairs

$$\widetilde{A} = \{\langle \mu_A(u), u \rangle\}, \tag{1}$$

where $\mu_A : U \to [0;1]$ is a mapping of the set U into the unit segment [0;1] which is called the membership function of the fuzzy set $\widetilde{A}$.

The value of the membership function $\mu_A(u)$ for the element $u \in U$ is called the dependence degree. For simplicity expression (1) is considered to be equivalent to the following ones:

$$A = \bigcup_{u \in U} \mu_A(U) \Big| u = \bigcup_{u \in U} \mu_u^A \Big| u = \bigcup_{u \in U} \mu_u \Big| u.$$

Consistently introducing the concept of fuzzyness, in the opinion of L.Zadeh [7; 8], it is possible to receive fuzzy analogues of all the basic mathematical concepts and to create the necessary formal device for modeling human judgements. Besides at some stage of construction of fractal object, concepts of fuzzyness and fractalness have much in common.

In their works Zadeh [7; 8], Klement [9] and Hohle [10] studied clear-valued and fuzzy-valued fuzzy measure. These works allow to introduce the concept of clear-valued and fuzzy-valued fuzzy fractal.

Definition 1. The clear set X consisting of all the elements of the fuzzy set $\widetilde{X}$ with membership function $\mu(x) > 0$ is called support of $\widetilde{X}$.

Definition 2. The fussy set $\widetilde{x}_i \subset \widetilde{X}$ such as

$$\mu_{\widetilde{x}_i}(x) = \text{const}, \forall x \in \widetilde{x},$$

is called the sphere of the fuzzy set $\widetilde{X}$.

Definition 3. The number $\mu_i \cdot \widetilde{m}(\widetilde{A}_i)$, where μ_i is the membership function of the i-th sphere $(\widetilde{m}(\widetilde{A}_i))$ and $\widetilde{m}(\widetilde{A}_i)$ is the clear measure (Lebesgue or Hausdorff measure) of the i-th sphere support, is called g-reduced fuzzy measure of the i-th sphere.

This definition of the fuzzy measure does not contradict to the definition of Sygeno's fuzzy measure and his axioms [11].

For investigating fuzzy sets the obvious property

$$\widetilde{m}(\widetilde{A}) = \sum_i \mu_i \widetilde{m}(\widetilde{A}_i)$$

can be applied.

Definition 4. The ordered set (x, μ, g) is called the space of fuzzy measure.

In a general case the additivity

$$g(A \cup B) \neq g(A) + g(B)$$

does not take place for the fuzzy measure. It means that the fuzzy measure is an one-parameter expansion of measure probability: $g(\widetilde{A})$ characterizes the degree of fuzzyness of $\widetilde{A}$, that is estimation of fuzzyness of the expression "$x \in \widetilde{A}$".

Definition 5. The number

$$\widetilde{M}_\varphi^H(\widetilde{A}) = \sum_i \mu_i \overline{m}_H(\overline{A}_i),$$

where $\overline{m}_H(\overline{A}_i)$ is corresponding Hausdorff entropy φ-dimensional measure of the i-th sphere support, is called clear-valued fuzzy Hausdorff entropy φ-dimensional measure of the set $\widetilde{A}$.

Definition 6. The nonnegative number φ_0 such as

$\widetilde{M}_{\varphi}^{H}(\widetilde{A})=0, \forall\varphi>\varphi_0$

and

$\widetilde{M}_{\varphi}^{H}(\widetilde{A})=\infty, \forall\varphi<\varphi_0$,

is called clear-valued Hausdorff-Bezikovich dimension of the fuzzy set $\widetilde{A}$.

Definition 7. A fuzzy set for which clear-valued Hausdorff-Bezikovich dimension is a fractional number is called a clear-valued fractal in the narrow sense. In the case of construction of the fuzzy fractal theory on the basis of definition of clear-valued measure, fractalness of the fuzzy set is reduced to fractalness of its support (i.e. the clear set). Therefore it was suggested to use another approach to definition of fuzzy fractals – fuzzy-valued fuzzy fractals on the base of fuzzy measure.

The expanded binary arithmetic operation denoted by * for fuzzy numbers $\mu_A,\mu_B,\mu_C\in F(\Re)$, $\forall x,y,z\in\Re$ is defined as follows:

$$C=A*B\Leftrightarrow\mu_C(z)=\bigvee_{z=x*y}(\mu_A(x)\wedge\mu_B(y)).$$

A family of fuzzy subsets $\delta\subset[0,1]^*$ satisfying the following conditions:

$\varphi-const \;\Rightarrow\; \varphi\subset\delta$,

$\mu\in\delta \;\Rightarrow\; 1-\mu\in\delta$,

$(\mu_n)\in\delta^N \;\Rightarrow\; \bigvee_{n\in N}\mu_n\in\delta$

is called a fuzzy δ-algebra.

The pair (x,δ) is called the fuzzy measured space, and elements from δ are called fuzzy measured sets.

Fuzzy-valued fuzzy measure on (x,δ) is defined by Hohle [10] as the function $\widetilde{m}:\delta\to N(\overline{\Re}_+)$ such that

$\tau_\wedge(\widetilde{m}(\mu\vee\nu),\widetilde{m}(\mu\wedge\nu))=\tau_\wedge(\widetilde{m}(\mu),\widetilde{m}(\nu))$;

$(\mu_n)_n\in N\in\delta^n, \mu_n\le\mu_{n+1}$

$\Rightarrow\widetilde{m}(\bigvee_{n\in N}\mu_n)=\bigvee_{n\in N}\widetilde{m}(\mu_n)$,

where $N(\overline{\Re}_+)$ is the set of all the fuzzy nonnegative numbers, $\tau_\wedge$ is natural algebraic operation (can be considered by means of Zadeh's principle [8]).

Let's consider an arbitrary fuzzy set $\widetilde{A}$ and divide it into subsets $\widetilde{A}_i$ so that all elements of each subset have the same membership function. We have

$$\bigcup_i\widetilde{A}_i=\widetilde{A}, \bigcap_i\widetilde{A}_i=0.$$

Definition 8. The fuzzy number $\widetilde{\widetilde{m}}$ defined as follows:

$$\widetilde{\widetilde{m}}(\widetilde{A})=(m_1|\mu_1,m_2|\mu_2,m_3|\mu_3,\ldots,m_n|\mu_n),$$

is called the fuzzy-valued fuzzy measure of the set $\widetilde{A}$.

Here m_1 is the clear measure of the set A_1, m_2 is the clear measure of the set $A_2\cup A_1$, m_i is the clear measure of the set $\bigcup_i A_i$, $i=1,2,\ldots,n$.

In general case ordering relationship of the type "more", "less" etc. on the set of fuzzy numbers are fuzzy.

Only in the case when the intersection of supports of fuzzy numbers A and B is empty, relation between these numbers is clear. Fuzzy-valued fuzzy measure built in such a way corresponds to Hohle's definition.

For construction of fuzzy-valued fuzzy fractal it is possible to use fuzzy numbers $(L-R)$ of type [8] which have simpler interpretation of extended binary operations:

$L(-x)=L(x);\ R(-x)=R(x);\ L(0)=1=R(0)$,

where L and R are non-increasing functions on the set of nonnegative real numbers.

The system of sets $\{F_n\}$ is called to cover the set X if every point of X belongs at least to one of the sets of this system (fuzzy membership is meant). The system $\{F_n\}$ is called the covering of the set X.

By the $\varphi_{\widetilde{P}}$ – sphere we shall understand the fuzzy set of points of the space distant from some fixed point not more than the given number $\frac{\widetilde{P}}{2}$ (radius).

Its measure is assumed to be $V_\varphi=\widetilde{P}^\varphi$. If φ is constant then the measure of $\varphi_{\widetilde{P}}$ – sphere is independent of dimension of the space. The set $\widetilde{X}$ is said to be covered by $\varphi_{\widetilde{P}}$ – sphere if its diameter does not exceed the number $\widetilde{P}$.

Let $\widetilde{X}$ be bounded fuzzy set, φ be an arbitrary real number: $0\le\varphi<\infty$. For every $\widetilde{\varepsilon}>0$ we introduce the number $\widetilde{\widetilde{m}}_\varphi^\varepsilon(\widetilde{x})=\inf_{P<\widetilde{\varepsilon}}N_{\widetilde{X}}(\widetilde{P})\,\widetilde{P}^\varphi$, where $N_{\widetilde{X}}(\widetilde{P})$ is the least quantity of $\varphi_{\widetilde{P}}$ – spheres of diameter $\widetilde{P}$ necessary for covering the set $\widetilde{X}$. Infimum is over all possible $\widetilde{P}\le\widetilde{\varepsilon}$.

Definition 9. The number

$$\widetilde{\widetilde{m}}_\varphi(\widetilde{x})=\lim_{\varepsilon\to0}\widetilde{\widetilde{m}}_\varphi^\varepsilon(\widetilde{x})$$

is called the Hausdorff entropy φ-dimension fuzzy measure of a set $\widetilde{X}$

Definition 10. The real nonnegative number φ for which φ-dimension fuzzy measure of the set $\widetilde{X}$ is nontrivial is called the entropy Hausdorff-Bezikovich dimension of the fussy set $\widetilde{X}$.

Definition 11. A fuzzy set with fractional Hausdorff-Bezikovich dimension is called fuzzy-valued fuzzy fractal.

There exist clear field models of phenomena spreading in environments with considered characteristics. In the case of phenomena spreading without neutralization (element's protection, immunization) the field operator $\boldsymbol{\psi}(x)$ and its adjoint $\boldsymbol{\psi}^{+}(x)$ are introduced. They play respectively the roles of operators of annihilation and renewal for elements which perceived the phenomenon. If the first element that perceived the phenomenon at the initial moment was at the initial point of space-time $(x;t)=(0;0)$, then the density of perceived elements is represented by Green's function

$$\left\langle 0 \left| \boldsymbol{\psi}^{+}(x) e^{LT} \boldsymbol{\psi}(0) \right| 0 \right\rangle ,$$

where $L = \int \Lambda dx$ is Liouville's operator

$$L = \left[- D\nabla\boldsymbol{\psi}^{+}\nabla\boldsymbol{\psi} - b\boldsymbol{\psi} + \boldsymbol{\psi}\right] +$$
$$+ \left\{\alpha(1+\boldsymbol{\psi}^{+})\boldsymbol{\psi}^{+}\boldsymbol{\psi}\right\} - c(1+.\boldsymbol{\psi}^{+})\boldsymbol{\psi}^{+}\boldsymbol{\psi}^{2}.$$

Expression in brackets corresponds to diffusion and neutralization of the phenomenon, expression in parenthesis – to phenomenon spreading. The multiplier $(1+\boldsymbol{\psi}^{+})$ represents saturation due to finite density of elements, inclined to perceive the phenomenon.

Conclusions

The analysis of clear fractal models shows the possibility of widening models of dangerous environmental processes spreading on the basis of concepts of fuzzy fractal sets. The main problems of perspective developing are:

– fractal dimensions of clusters for repeated spreading of the same phenomenon;

– fractal dimensions of clusters for consequent spreading of different phenomena;

– fractal dimensions of clusters for simultaneous spreading of several phenomena;

– description of solutions of above problems in terms of clear-valued and fuzzy-valued fuzzy fractals.

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