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M. Fedorov

ON GENERAL SOLUTION OF ODE SYSTEM FOR MODELING STATIONARY COMPRESSIBLE FLOW OF PERFECT GAS WITH CONSTANT HEAT FLUX AND FRICTION IN THE CONSTANT-AREA CHANNEL

The possibility is demonstrated of taking a general solution of the differential equation system describing one-dimensional stationary flow of compressible gas with the given constant heat flux, friction and mass forces that can perform work in the constant-area channel.

It is well-known from heat and mass transfer theory the following system of differential equations describing one-dimensional stationary flow of compressible gas in the constant-area channel:

– continuity equation

$$\frac{d(\rho v)}{dx}=0; \tag{1}$$

– impulse equation

$$vdv+\frac{dP}{\rho}=f_x dx-\frac{2\tau_W}{\rho R_h}dx; \tag{2}$$

– energy equation

$$\frac{d}{dx}(c_P T+\frac{1}{2}v^2)=q_x+f_x=w_x, \tag{3}$$

where ρ – density; v – gas velocity; P – pressure; f_x – mass force; τ_W – tangent friction stress; R_h – channel hydraulic radius; c_P – specific heat at constant pressure; T – temperature; q_x – heat flux density referred to the mass flow rate unity.

Since a constant area channel is considered then

$$G=\rho vF,$$

where F is a cross section area.

Velocity and density may be expressed through pressure P, flow rate G and temperature T as follows

$$\rho=\frac{P}{gRT};$$

$$v=\frac{G}{\rho F}=\frac{GRgT}{PF}; \tag{4}$$

$$\frac{dv}{dx}=g\frac{GR}{F}\left(\frac{1}{P}\frac{dT}{dx}-\frac{T}{P^2}\frac{dP}{dx}\right),$$

where g – acceleration due to gravity; R – gas constant (for air R=29,27); G – gas flow rate.

Formulae for q_x and τ_W have the following form

$$q_x=\frac{Q}{LG};$$

$$\tau_W=\frac{1}{2\omega}\lambda\rho v^2,$$

where Q – heat flux; $\omega=4$; λ – loss factor.

Let's substitute formulae (4) into the energy equation (3) to get

$$c_P\frac{dT}{dx}+\left(\frac{gRG}{F}\right)^2\frac{T}{P}\left(\frac{1}{P}\frac{dT}{dx}-\frac{T}{P^2}\frac{dP}{dx}\right)=w_x,$$

collecting similar terms we have

$$\left[c_P+\left(\frac{gRG}{F}\right)^2\frac{T}{P^2}\right]\frac{dT}{dx}=w_x+\left(\frac{gRG}{F}\right)^2\frac{T^2}{P^3}\frac{dP}{dx}. \tag{5}$$

Now we transform the impulse equation, performing substitutions formulae (4) in it and the expression for τ_W

$$\frac{G}{F}\frac{gRG}{F}\left(\frac{1}{P}\frac{dT}{dx}-\frac{T}{P^2}\frac{dP}{dx}\right)+\frac{dP}{dx}=-\frac{2\lambda}{\omega D_h}\left(\frac{G}{F}\right)^2\frac{gRT}{P}+\frac{P}{gRT}f_x .$$

Collecting similar terms we have

$$\left(\frac{F^2}{gRG^2}-\frac{T}{P^2}\right)\frac{dP}{dx}=-\frac{1}{P}\frac{dT}{dx}-\frac{2\lambda}{\omega D_h}\frac{T}{P}+\left(\frac{F}{gRG}\right)^2\frac{P}{T}f_x . \tag{6}$$

Expressing from (5) a temperature gradient

$$\frac{dT}{dx}=\frac{w_x+\left(\frac{gRG}{F}\right)^2\frac{T^2}{P^3}\frac{dP}{dx}}{\left[c_P+\left(\frac{gRG}{F}\right)^2\frac{T}{P^2}\right]} \tag{7}$$

and substituting it into (6) we get

$$\left(\frac{F^2}{gRG^2}-\frac{T}{P^2}\right)\frac{dP}{dx}=-\frac{1}{P}\frac{w_x+\left(\frac{gRG}{F}\right)^2\frac{T^2}{P^3}\frac{dP}{dx}}{\left[c_p+\left(\frac{gRG}{F}\right)^2\frac{T}{P^2}\right]}-\frac{2\lambda}{\omega D_h}\frac{T}{P}+\left(\frac{F}{gRG}\right)^2\frac{P}{T}f_x .$$

With algebra the last equation is reduced to

$$\frac{dP}{dx}=\frac{-\frac{1}{P}w_x-\frac{2\lambda}{\omega D_h}\frac{T}{P}\left[c_p+\left(\frac{gRG}{F}\right)^2\frac{T}{P^2}\right]+\left(\frac{F}{gRG}\right)^2 f_x\frac{P}{T}\left[c_p+\left(\frac{gRG}{F}\right)^2\frac{T}{P^2}\right]}{\left\{\left[c_p+\left(\frac{gRG}{F}\right)^2\frac{T}{P^2}\right]\left(\frac{F^2}{gRG^2}-\frac{T}{P^2}\right)+\frac{1}{P}\left(\frac{gRG}{F}\right)^2\frac{T^2}{P^3}\right\}}. \tag{8}$$

The dominator can be transformed to be

$$\left\{\left[c_p+\left(\frac{gRG}{F}\right)^2\frac{T}{P^2}\right]\left(\frac{F^2}{gRG^2}-\frac{T}{P^2}\right)+\frac{1}{P}\left(\frac{gRG}{F}\right)^2\frac{T^2}{P^3}\right\}=$$

$$=c_p\left(\frac{F^2}{gRG^2}-\frac{T}{P^2}\right)+\left(\frac{gRG}{F}\right)^2\frac{T}{P^2}\left(\frac{F^2}{gRG^2}-\frac{T}{P^2}\right)+\frac{1}{P}\left(\frac{gRG}{F}\right)^2\frac{T^2}{P^3}.$$

Making further reductions the dominator takes the following final form

$$c_p\left(\frac{F^2}{gRG^2}-\frac{T}{P^2}\right)+\left(\frac{gRG}{F}\right)^2\frac{T}{P^2}\left(\frac{F^2}{gRG^2}-\frac{T}{P^2}\right)+\left(\frac{gRG}{F}\right)^2\frac{T^2}{P^4}=c_p\left(\frac{F^2}{gRG^2}-\frac{T}{P^2}\right)+$$
$$+\left(\frac{gRG}{F}\right)^2\frac{F^2}{gRG^2}\frac{T}{P^2}-\left(\frac{gRG}{F}\right)^2\frac{T}{P^4}+\left(\frac{gRG}{F}\right)^2\frac{T^2}{P^4}=c_p\frac{F^2}{gRG^2}+\left(gR-c_p\right)\frac{T}{P^2}. \tag{9}$$

Substituting the dominator (9) in (8) and, in turn, (8) in (7) we have

$$\frac{dT}{dx}=\frac{w_x}{\left[c_p+\left(\frac{gRG}{F}\right)^2\frac{T}{P^2}\right]}+$$

$$+\frac{\left(\frac{gRG}{F}\right)^2\frac{T^2}{P^3}\left\{-\frac{1}{P}w_x-\frac{2\lambda}{\omega D_{\text{h}}}\frac{T}{P}\left[c_p+\left(\frac{gRG}{F}\right)^2\frac{T}{P^2}\right]+\left(\frac{F}{gRG}\right)^2 f_x\frac{P}{T}\left[c_p+\left(\frac{gRG}{F}\right)^2\frac{T}{P^2}\right]\right\}}{\left[c_p+\left(\frac{gRG}{F}\right)^2\frac{T}{P^2}\right]\left[c_p\frac{F^2}{gRG^2}+\left(gR-c_p\right)\frac{T}{P^2}\right]}.$$

The nominator of the previous equation can be reduced to the form

$$\left[c_p\frac{F^2}{gRG^2}+\left(gR-c_p\right)\frac{T}{P^2}-\left(\frac{gRG}{F}\right)^2\frac{T^2}{P^4}\right]w_x-\frac{2\lambda}{\omega D_{\text{h}}}\left(\frac{gRG}{F}\right)^2\frac{T^3}{P^4}\left[c_p+\left(\frac{gRG}{F}\right)^2\frac{T}{P^2}\right]+$$

$$+f_x\frac{T}{P^2}\left[c_p+\left(\frac{gRG}{F}\right)^2\frac{T}{P^2}\right]=\left\{\frac{F^2}{gRG^2}\left[c_p+\left(\frac{gRG}{F}\right)^2\frac{T}{P^2}\right]-\left[c_p+\left(\frac{gRG}{F}\right)^2\frac{T}{P^2}\right]\frac{T}{P^2}\right\}w_x-$$

$$-\frac{2\lambda}{\omega D_{\text{h}}}\left(\frac{gRG}{F}\right)^2\frac{T^3}{P^4}\left[c_p+\left(\frac{gRG}{F}\right)^2\frac{T}{P^2}\right]+f_x\frac{T}{P^2}\left[c_p+\left(\frac{gRG}{F}\right)^2\frac{T}{P^2}\right].$$

As the result, we have the following autonomous ODE system for calculation of channel flow with friction, heat transfer and mass forces that can perform work

$$\frac{dP}{dx}=\frac{-\frac{1}{P}w_x-\frac{2\lambda}{\omega D_{\text{h}}}\frac{T}{P}\left[c_p+\left(\frac{gRG}{F}\right)^2\frac{T}{P^2}\right]+\left(\frac{F}{gRG}\right)^2 f_x\frac{P}{T}\left[c_p+\left(\frac{gRG}{F}\right)^2\frac{T}{P^2}\right]}{\left[c_p\frac{F^2}{gRG^2}+\left(gR-c_p\right)\frac{T}{P^2}\right]};\qquad(10)$$

$$\frac{dT}{dx}=\frac{\left[\frac{F^2}{gRG^2}-\frac{T}{P^2}\right]w_x-\frac{2\lambda}{\omega D_{\text{h}}}\left(\frac{gRG}{F}\right)^2\frac{T^3}{P^4}+f_x\frac{T}{P^2}}{\left[c_p\frac{F^2}{gRG^2}+\left(gR-c_p\right)\frac{T}{P^2}\right]}.\qquad(11)$$

Dividing equation (11) to (10), we have

$$\frac{dT}{dP}=-\frac{\left[\frac{F^2}{gRG^2}-\frac{T}{P^2}\right]w_x-\frac{2\lambda}{\omega D_{\text{h}}}\left(\frac{gRG}{F}\right)^2\frac{T^3}{P^4}+f_x\frac{T}{P^2}}{\frac{1}{P}w_x+\frac{2\lambda}{\omega D_{\text{h}}}\frac{T}{P}\left[c_p+\left(\frac{gRG}{F}\right)^2\frac{T}{P^2}\right]-\left(\frac{F}{gRG}\right)^2 f_x\frac{P}{T}\left[c_p+\left(\frac{gRG}{F}\right)^2\frac{T}{P^2}\right]}.\qquad(12)$$

Equation (12) groups together pressure and temperature at every flow section, whence, if it has been possible to solve, it would be possible to eliminate one unknown i.e. to reduce the problem to solving the single differential equation.

Making the following change of variables

$$z=\frac{T}{P};$$

$$T=zP;$$

$$\frac{dT}{dP} = P\frac{dz}{dP} + z\,,$$

we have

$$P\frac{dz}{dP} + z = -\frac{P\left[\frac{F^2}{gRG^2} - \frac{z}{P}\right]w_x - \frac{2\lambda}{\omega D_\mathrm{h}}\left(\frac{gRG}{F}\right)^2 z^3 + f_x z}{w_x + \frac{2\lambda}{\omega D_\mathrm{h}} zP\left[c_p + \left(\frac{gRG}{F}\right)^2 \frac{z}{P}\right] - \left(\frac{F}{gRG}\right)^2 f_x \frac{P}{z}\left[c_p + \left(\frac{gRG}{F}\right)^2 \frac{z}{P}\right]}.$$

Making some reductions in the previous equation, we get

$$\left\{w_x + \frac{2\lambda}{\omega D_\mathrm{h}} zP\left[c_p + \left(\frac{gRG}{F}\right)^2 \frac{z}{P}\right] - \left(\frac{F}{gRG}\right)^2 f_x \frac{P}{z}\left[c_p + \left(\frac{gRG}{F}\right)^2 \frac{z}{P}\right]\right\}\frac{dz}{dP} =$$

$$= -\frac{F^2}{gRG^2} w_x - c_p \frac{2\lambda}{\omega D_\mathrm{h}} z^2 + c_p\left(\frac{F}{gRG}\right)^2 f_x\,.$$

Making independent variables to be dependent and vice versa, we have

$$\frac{dP}{dz} = \frac{w_x + \frac{2\lambda}{\omega D_\mathrm{h}} zP\left[c_p + \left(\frac{gRG}{F}\right)^2 \frac{z}{P}\right] - \left(\frac{F}{gRG}\right)^2 f_x \frac{P}{z}\left[c_p + \left(\frac{gRG}{F}\right)^2 \frac{z}{P}\right]}{\left[c_p\left(\frac{F}{gRG}\right)^2 f_x - \frac{F^2}{gRG^2} w_x\right] - c_p \frac{2\lambda}{\omega D_\mathrm{h}} z^2}.$$

Doing simple manipulations with the previous equation, it can be reduced to the form

$$\frac{dP}{dz} - \frac{c_p \frac{2\lambda}{\omega D_\mathrm{h}} z - c_p\left(\frac{F}{gRG}\right)^2 f_x \frac{1}{z}}{\left\{\left[c_p\left(\frac{F}{gRG}\right)^2 f_x - \frac{F^2 w_x}{gRG^2}\right] - c_p \frac{2\lambda}{\omega D_\mathrm{h}} z^2\right\}} P = \frac{w_x + \frac{2\lambda}{\omega D_\mathrm{h}}\left(\frac{gRG}{F}\right)^2 z^2 - f_x}{\left\{\left[c_p\left(\frac{F}{gRG}\right)^2 f_x - \frac{F^2 w_x}{gRG^2}\right] - c_p \frac{2\lambda}{\omega D_\mathrm{h}} z^2\right\}}.$$

Introducing for similarity auxiliary variables it is evident that the above ODE is a linear one of the form

$$\frac{dP}{dz} + R(z)P = Q(z)\,.$$

It is known that such a linear heterogeneous ODE has a solution in the form

$$P = e^{-\int R(z)dz}\left[\int Q(z)e^{\int R(z)dz} dz + \mathrm{const}\right], \tag{13}$$

where

$$R(z) = -\frac{c_p \frac{2\lambda}{\omega D_\mathrm{h}} z - c_p\left(\frac{F}{gRG}\right)^2 f_x \frac{1}{z}}{\left\{\left[c_p\left(\frac{F}{gRG}\right)^2 f_x - \frac{F^2}{gRG^2} w_x\right] - c_p \frac{2\lambda}{\omega D_\mathrm{h}} z^2\right\}};$$

$$Q(z)=\frac{w_x+\frac{2\lambda}{\omega D_h}\left(\frac{gRG}{F}\right)^2 z^2-f_x}{\left\{\left[c_p\left(\frac{F}{gRG}\right)^2 f_x-\frac{F^2}{gRG^2}w_x\right]-c_p\frac{2\lambda}{\omega D_h}z^2\right\}}.$$

Equation (13) is an integral of energy of one-dimensional stationary flow of compressible perfect gas with friction, heat transfer and mass forces that can perform work in the constant-area channel.

We demonstrate below, that for the given problem (1)–(3) or (10)–(11) the general solution may be taken. For this purpose we differentiate (13) in x

$$\frac{dP}{dx}=\left[\int Q(z)\, e^{\int R(z)dz}dz+\text{const}\right]\left(e^{-\int R(z)dz}\right)'\frac{dz}{dx}+e^{-\int R(z)dz}\left[\int Q(z)\, e^{\int R(z)dz}dz+\text{const}\right]'\frac{dz}{dx}.$$

Finally, the above derivative may be rewritten as follows

$$\frac{dP}{dx}=\left[\int Q(z)\, e^{\int R(z)dz}dz+\text{const}\right]\left(\int R(z)dz\right)\left(\int\frac{\partial R(z)}{\partial x}dz\right)e^{-\int R(z)dz}\frac{dz}{dx}+$$

$$+e^{-\int R(z)dz}\left[\int\frac{\partial Q(z)}{\partial x}\, e^{\int R(z)dz}dz+\int Q(z)\left(\int R(z)dz\right)\left(\int\frac{\partial R(z)}{\partial x}e^{-\int R(z)dz}dz\right)dz\right]\frac{dz}{dx}.$$

Further, we introduce variable z in equation (10), then

$$\frac{dP}{dx}=-\frac{1}{gR}\left(\frac{gRG}{F}\right)^2\frac{w_x+\frac{2\lambda}{\omega D_h}zz\left[c_pP+\left(\frac{gRG}{F}\right)^2 z\right]-\left(\frac{F}{gRG}\right)^2 f_x\frac{1}{z}\left[c_pP+\left(\frac{gRG}{F}\right)^2 z\right]}{\left[c_pP+\left(\frac{gRG}{F}\right)^2\left(1-\frac{c_p}{gR}\right)z\right]}.$$

We substitute the derivative obtained and (13) in the previous equation to get

$$\left\{\left[\int Q(z)e^{\int R(z)dz}dz+\text{const}\right]\left(\int R(z)dz\right)\left(\int\frac{\partial R(z)}{\partial x}dz\right)e^{-\int R(z)dz}+\right.$$

$$\left.+e^{-\int R(z)dz}\left[\int\frac{\partial Q(z)}{\partial x}\, e^{\int R(z)dz}dz+\int Q(z)\left(\int R(z)\cdot dz\right)\left(\int\frac{\partial R(z)}{\partial x}e^{-\int R(z)dz}dz\right)dz\right]\right\}\frac{dz}{dx}=$$

$$=-\frac{1}{gR}\left(\frac{gRG}{F}\right)^2\frac{w_x+\left\{c_pe^{\int R(z)dz}\left[\int Q(z)e^{-\int R(z)dz}dz+\text{const}\right]+\left(\frac{gRG}{F}\right)^2 z\right\}\left(\frac{2\lambda}{\omega D_h}z-f_x\frac{1}{z}\right)}{\left[c_pe^{\int R(z)dz}\left[\int Q(z)e^{-\int R(z)dz}dz+\text{const}\right]+\left(1-\frac{c_p}{gR}\right)\left(\frac{gRG}{F}\right)^2 z\right]}.$$

It is evident, that the above ODE variables are separable, consequently, it is demonstrated that for the one-dimensional stationary flow model (1)–(3) of compressible perfect gas with friction, heat transfer and mass forces that can perform work in the constant-area its general solution exists.

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