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DETERMINATION OF THE INDEX OF CLEARNESS FOR THE DISTRIBUTION OF FUZZY MEASURES

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Abstract

Objective: *To justify the need and propose a new clarity index for the distribution of a fuzzy measure with an arbitrary modality. Conduct a study of the new index and show its effectiveness, sensitivity and ease of use for analyzing fuzzy data.* ***Methods:*** *for solving the problem using methods of set theory, fuzzy measures theory and functional analysis, and formal logic.* ***Results:*** *A new index is justified and proposed, which provides an estimate of the clarity for the distribution of a fuzzy measure with an arbitrary modality. Formulas are proposed for calculating the clarity index for a fuzzy measure on a discrete and continuous space. It is proved that the proposed index satisfies the properties that are advanced to the clarity indices. Additional dependencies are obtained to calculate the clarity index based on the use of level sets for the fuzzy measure density function.* ***Discussion:*** *the results of the calculation of the clarity index for a family of fuzzy measures with various modalities are presented. It is shown that the proposed index completely satisfies the advanced requirements to the logic of the performance of the clarity index and takes into account the modality of the fuzzy measure.*

Keywords: Fuzzy measure; index of clearness; quality of control; sets

1. Introduction

This material is devoted to the definition of the clearness indicator of fuzzy measures distribution. Formulation of the question of this study is dictated by the fact that to date, as the analysis of research shows, there is no corresponding index. The need to introduce this indicator of clearness for use in control problems under uncertainty has been discussed and argued in a number of papers [1,2]. But the introduction of clarity measures or duals to them fuzzy measures of confidence distribution was oriented, generally, on the analysis of the membership function distribution of a fuzzy set

It is known that the membership function of a fuzzy set in the case of focal elements nesting when constructing of fuzzy measures coincide with the distribution of the density of fuzzy possibility measures [3]. This measure is a special case of fuzzy measures, which can be used in solving control problems under uncertainty. In particular, Higashi-Klier fuzziness measures [4], Jager clarity measure [5], a number of fuzzy measures that are based on the metric approach [1] using a variety of metrics (eg, Hamming metric, Euclidean, Steinhaus, Tanimoto and other) [6] have been proposed.

However, the above measures and indicators of clarity (fuzziness) do not allow to estimate the clarity degree of an arbitrary fuzzy measures density distribution with the modality, different from the modality of "possible". At the same time the introduction of such a measure is necessary to address the control problems under uncertainty.

For example, to determine the performance of the fuzzy filter [7, 8] in the height control loop of UAV flight In this case, the estimated density distribution of the output signal fuzzy measure. In this case the estimated density distribution of fuzzy measure of the output signal during the filter operation must ensure clarity maximization of output estimating density distribution of fuzzy measures in the area of the true signal values.

Equally important is the use of the clarity index of estimation density distribution of fuzzy measures for the control algorithms of extrapolation and prediction. In particular, to solve these problems the proposed algorithms should ensure on the one hand clarity maximizing of output estimation distribution density of fuzzy measure, on the other hand, true signal getting into the area with the highest density of fuzzy measures [9].

In such a way, in practical problems solving under uncertainty is expected widespread use of clarity index of fuzzy measure density distribution. However, current approaches do not allow to obtain an clarity estimation of the fuzzy measures distribution of the arbitrary modality.

2. Formulation of the research problem

To address the issue we take the distribution of fuzzy measures $g_X(\cdot): 2^X \to [0,1]$ on a finite space X. In the future, we will consider a discrete as well as continuous case for space X. Given the fact that to date the Sugeno fuzzy measures [10] are the most widely used in solving practical problems using fuzzy measures, we will rely on this kind of fuzzy measures in our studies. Note that the clarity index proposed below can be used for other types of fuzzy measures, such as Tsukamoto measures [11].

It is known that fuzzy measure satisfies the following axioms [12]:

1. Restrictions: $g_X(\varnothing)=0,\ g_X(X)=1$.
2. Monotony: $\forall A,B\subseteq X,\ A\subseteq B$ the condition $g_X(A)\le g_X(B)$.
3. Continuity $\forall F_n\subseteq X$, $\{F_n\}$ - monotonically increasing or decreasing sequence of subsets following condition is satisfied: $g_X\left(\lim_{n\to\infty}F_n\right)=\lim_{n\to\infty}g_X(F_n)$.

Let fuzzy measure $g_X(\cdot):2^X\to[0,1]$ has an arbitrary modality and satisfies Sugeno measures λ-rule, for which the following condition is satisfied: $\forall A,B\subseteq X,\ A\cap B=\varnothing,\ \lambda\in\left[-1,+\infty\right[$.

$$g(A\cup B)=g(A)+g(B)+\lambda\cdot g(A)\cdot g(B). \quad (1)$$

Let us denote $h(x):X\to[0,1]$ the density distribution of fuzzy measure. For a given fuzzy measure, we need to identify some clarity indicator of measures distribution, which would allow to estimate the level of uncertainty inherent in the distribution of measure $g_X(\cdot):2^X\to[0,1]$. At the same time we have to take into account the property that if the measure is concentrated at one point $x^*\in X$, then the measure is a measure of Dirac [13] and it is as clear as possible. In this case, the clarity indicator should strive for the maximum value. Another limiting case is when we have a complete uncertainty, that is, when the condition $\forall x\in X,\ h(x)=const$ is satisfied. In this case, clarity index of fuzzy measure distribution should strive to minimum. It is natural to assume that in the case of a uniform distribution of fuzzy measure density function the clarity indicator should be minimal at any modality of measure. These considerations correspond to the intuitive understanding of the clarity of fuzzy measure distribution. This fact can be expressed as the following statement:

"The more smaller subspace (smaller area) fuzzy measure is concentrated, the more accurate it is and the higher the clarity indicator. Conversely, the more uniformly in the space X the fuzzy measure of an arbitrary modality is represented, the more uncertain it is, and the lower the clarity indicator."

In this situation it is necessary to make one remark. Existing clarity indicators really consider this condition for fuzzy sets. However, to apply the structure of these indicators to the fuzzy measure is difficult due to the fact that the fuzzy measure, in contrast to the membership function of a fuzzy set is a function of set, which assigns to each subset of the set X some value in the unit interval. Not accounting of this property leads to the estimation error of measure degree clarity distribution when attempting to use existing indicators clarity.

3. The purpose of the study

Thus, the objective of our study is to determine the clarity index for the distribution of Sugeno fuzzy measure of an arbitrary modality, that satisfies the following properties:

1. If fuzzy measure $g_X(\cdot):2^X\to[0,1]$ satisfies the relationship $g_X(\cdot)=\Delta_a(\cdot)$, where $\Delta_a(\cdot)$ - Dirac measure, concentrated at the point $a\in X$, then the clarity indicator $J(g_X)$ tends to the maximum value.
2. If $g_X(\cdot):2^X\to[0,1]$ has a uniform measure distribution density $\forall x\in X,\ h(x)=const$ then the clarity indicator $J(g_X)$ tends to the minimum value.
3. For two fuzzy measures $g_X^1(\cdot)$ and $g_X^2(\cdot)$, specified in space X if the following condition satisfied $\forall A,B\subseteq X,\quad g_X^1(A)\ge g_X^2(B)$, $Card(A)\le Card(B)$ then $J(g_X^1)\ge J(g_X^2)$.

4. Research Methods

For the study, we will use the approaches of the set theory, theory of fuzzy measure, functional analysis and formal logic.

5. Research results

In the beginning, we consider the Clarity measure of fuzzy measure distribution of an arbitrary modality. To construct the clarity indicator let's consider on a discrete space X confidence fuzzy measure $g_X(\cdot):2^X\to[0,1]$ with non-decreasing on $x\in X$ the fuzzy measure density function $\forall x,y\in X,\ x\le y,\quad h(x)\le h(y)\in[0,1]$ (Fig. 1).

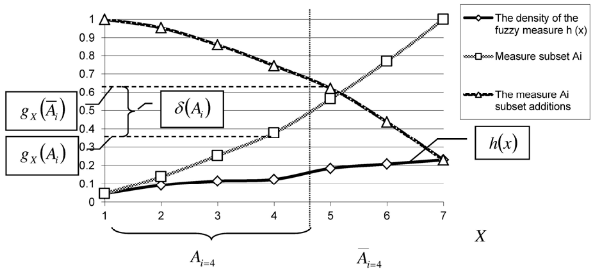


Fig. 1. Explanation to the determination of clarity index for the distribution of fuzzy measures.

Any point $x_i \in X$ divides all set X on two mutually complementary subsets so that $\forall x_i \in X,\ A_i \cap \overline{A}_i = \varnothing,\ A_i \cup \overline{A}_i = X$. We define an increasing sequence of subsets $\{A_i\},\ A_i = [x_0, x_i] \subseteq X$, $A_0 = \varnothing$. For a discrete set X index $i = I$ defined by the set of natural numbers $I \subseteq N$ Based on the conditions of additions, the sequence of sets $\{\overline{A}_i\}$ will be decreasing sequence.

The value $g_X(A_i)$ will define "volume of measure" $g_X(\cdot)$, concentrated in a subset $A_i \subseteq X$. At the same time, based on the properties of the monotony of fuzzy measures we have $\forall A_i \subseteq X$, $g_X(A_i) \le g_X(A_{i+1})$. Thus, the function $g_X(A_i): 2^X \to [0,1]$ will be a non-decreasing function of each $A_i \subseteq X$. Supplement measure $g_X(\overline{A}_i)$ based on the monotony property of fuzzy measure for the sequence $\{A_i\}$ will be non-increasing function for each $A_i \subseteq X$. It will determine the degree of fuzzy measure $g_X(\cdot)$ concentration in the area $\overline{A}_i \subseteq X$.

Consider the function $\delta(x_i): X \to [-1,1]$ in the form:

$$\delta(x_i) = g_X(\overline{A}_i) - g_X(A_i). \qquad (2)$$

This function shows how much more the measure $g_X(\cdot)$ is concentrated in the subset $\overline{A}_i \subseteq X$ than in supplements thereto A_i.

For shown in (Fig. 1) fuzzy measure with a density of $h(x): X \to [0,1]$, function $\delta(x_i)$ taking into account the empty set ($i = 0,\ A_0 = \varnothing$) will have the form shown in (Fig. 2).

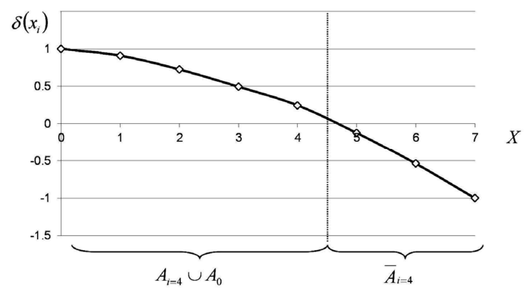
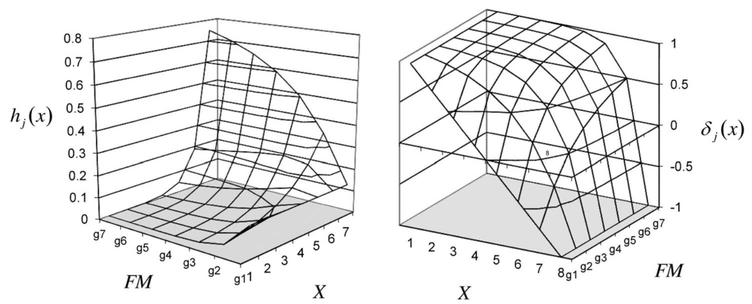


Fig. 2. Function $\delta(x_i)$.

It is easy to see that $\delta(x_i)$ in general, is not linear, and defines the region in space X , where Supplement measure $\overline{A}_i \subseteq X$ exceeds the measure of subset $A_i \subseteq X$. In (Fig. 3) as an example the dependences of the density distributions of fuzzy measures $\{h_j(x) | j = \overline{1,7}\}$ for a family of fuzzy confidence measures $FM = \{g_j(\cdot) | j = \overline{1,7}\}$ and their respective functions $\delta_j(x)$ are presented.



a *b*

Fig. 3. Density distribution of fuzzy measures a) and their corresponding functions $\delta(x_i)$ b).

The ratio of the areas shown in (Fig. 2) for positive and negative values of the function $\delta(x_i): X \to [-1,1]$ could be used to determine a clarity index of fuzzy measure distribution. Function $\delta(x_i)$ for a fixed $x_i \in X$ shows how much the measure concentrated in the subset $\overline{A}_i \subseteq X$ then in the subset $A_i \subseteq X$.

If we consider all subsets $A_i \subseteq X$ than the clarity indicator of measure distribution may be represented as the integral relationship:

$$J(g_X) = \frac{1}{Card(X)} \cdot \int_X \delta(x)dx . \quad (3)$$

where $Card(X)$ - cardinality X, the use of which in the expression for the clarity index provides a normalization of index $J(g_X)$ to 1.

Practically, the index $J(g_X)$ shows how much more the measure $g_X(\cdot): 2^X \to [0,1]$ concentrated in the area at large values $x_i \in X$, than at small. It should be noted that the indicator $J(g_X)$ can not take a negative value as a function of measure density $h(x): X \to [0,1]$ is a nondecreasing function on the space X. Taking into account the easy transition from the integral sign to the sign of the sum for the discrete set X clarity indicator of fuzzy measure distribution can be represented as:

$$J(g_X) = \frac{1}{I-1} \cdot \sum_{i=\overline{0,I}} \delta(x_i) . \quad (4)$$

where $I = Card(X)$.

Let us check whether the requirements for determining the distribution of the fuzzy measure clarity indicator are carried out. In particular, we consider two limiting cases mentioned above:

a. clear distribution of fuzzy measure, given in the form of the Dirac measure on the space;
b. uniform distribution of fuzzy measure on the whole space.

Lemma 1. For crisp distribution of fuzzy measure $g_X(\cdot)$, concentrated at a point $a \in X$ clarity indicator for this measure will be $J(g_X) = 1$.

Proof. At absolutely clear measure $g_X(\cdot)$, concentrated at a point $a \in X$, have:

$$\forall x \in X, \quad g_X(\{x\}) = h(x) = \begin{cases} 1, x = a; \\ 0, x \neq a. \end{cases}$$

In this case, the function $\delta(x): X \to [-1,1]$ will be:

$$\forall x \in X, \quad \delta(x) = \begin{cases} 1, x \neq a; \\ -1, x = a. \end{cases}$$

Let ε point neighborhood $a \in X$. Then:

$$J(g_X) = \frac{1}{Card(X)} \cdot \int_X \delta(x)dx =$$

$$= \frac{1}{Card(X)} \cdot \left(\int_{X \setminus [a-\varepsilon, a+\varepsilon]} \delta(x)dx + \int_{[a-\varepsilon, a+\varepsilon]} \delta(x)dx \right) =$$

$$= \frac{Card(X \setminus [a-\varepsilon, a+\varepsilon])}{Card(X)} - \frac{Card([a-\varepsilon, a+\varepsilon])}{Card(X)}.$$

At $\varepsilon \to 0$ $Card(X \setminus [a-\varepsilon, a+\varepsilon]) \to 1$, and $Card([a-\varepsilon, a+\varepsilon]) \to 0$ and consequently $J(g_X) \to 1$. For the discrete case, we have a similar result:

$$J(g_X) = \frac{1}{I-1} \cdot \sum_{i=\overline{0,I}} \delta(x_i) =$$

$$= \frac{1}{I-1} \cdot \left[\sum_{i=\overline{1,I-1}} \delta(x_i) + (\delta(x_0) + \delta(x_I)) \right].$$

Based on the expression (2) we have $\delta(x_0) = 1$, $\delta(x_I) = -1$ Then the clarity indicator will be:

$$J(g_X) = \frac{1}{I-1} \cdot \sum_{i=\overline{0,I}} \delta(x_i) =$$

$$= \frac{1}{I-1} \cdot \sum_{i=\overline{1,I-1}} \delta(x_i) = \frac{1}{I-1} \cdot (I-1) = 1.$$

Thus, when the exact distribution of fuzzy measure (the measure is concentrated at a single point of space) the clarity indicator acquires value equal to 1.

Lemma 2. For a fully uncertain distribution of fuzzy measure $g_X(\cdot)$, for which the density of fuzzy measure acquires the values $\forall x \in X, h(x) = const$ clarity indicator will be $J(g_X) = 0$.

Proof. To illustrate the proof we consider a discrete case of fuzzy measure distribution. Measure of subset $A_i \subseteq X$ consisting of i points is given by [1]:

$$g_X(A_i) = \frac{1}{\lambda} \cdot \left(\prod_{x_i \in A_i} (1 + \lambda \cdot h(x_i)) - 1 \right). \quad (5)$$

For uniform distribution of fuzzy measure following relation holds:

$$g_X(A_i) = \frac{1}{\lambda} \cdot \left((1 + \lambda \cdot g)^i - 1 \right),$$

where $\forall x_i \in X, \; h(x_i) = g$. Similarly, we represent the measure value for complement $\overline{A_i} = X \setminus A_i$ in the form:

$$g_X(\overline{A}_i) = \frac{1}{\lambda}\cdot\left((1+\lambda\cdot g)^{I-i} - 1\right).$$

Based on introduced notations the function $\delta(x_i)$ for point $x_i \in X$ will be:

$$\delta(x_i) = g_X(\overline{A}_i) - g_X(A_i) =$$
$$= \frac{1}{\lambda}\cdot\left((1+\lambda\cdot g)^{I-i} - 1\right) - \frac{1}{\lambda}\cdot\left((1+\lambda\cdot g)^{i} - 1\right) =$$
$$= \frac{1}{\lambda}\cdot\left((1+\lambda\cdot g)^{I-i} - (1+\lambda\cdot g)^{i}\right).$$

Then, the clarity index can be written as:

$$J(g_X) = \frac{1}{(I-1)\cdot\lambda}\cdot\sum_{i=\overline{0,I}}\left((1+\lambda\cdot g)^{I-i} - (1+\lambda\cdot g)^{i}\right) =$$
$$= \frac{1}{(I-1)\cdot\lambda}\cdot\sum_{i=\overline{0,I}}\left(a^{I-i} - a^{i}\right),$$

where $a = (1+\lambda\cdot g)$.

Let us consider the resulting expression. Value $i = \overline{0,I}$. Hence for odd I we can regroup the values under the sign of the sum as follows:

$$J(g_X) = \frac{1}{(I-1)\cdot\lambda}\cdot\sum_{i=\overline{0,I}}\left[\left(a^{I-i} - a^{i}\right) + \left(a^{i} - a^{I-i}\right)\right]\cdot\frac{1}{2} =$$
$$= \frac{1}{(I-1)\cdot\lambda}\cdot 0 = 0.$$

If I - is even the expression takes the form:

$$J(g_X) = \frac{1}{(I-1)\cdot\lambda}\cdot\left(a^{I-k} - a^{k}\right) +$$
$$+ \frac{1}{(I-1)\cdot\lambda}\cdot\sum_{\substack{i=\overline{0,I}\\ i\neq k}}\left[\left(a^{I-i} - a^{i}\right) + \left(a^{i} - a^{I-i}\right)\right]\cdot\frac{1}{2} =$$
$$= \frac{1}{(I-1)\cdot\lambda}\cdot\left\{\left(a^{I-k} - a^{k}\right) + 0\right\}.$$

Where k - is the value at which $I-k=k$. But at this case $a^{I-k} = a^{k}$ consequently $J(g_X) = 0$.

In such a way, with a uniform distribution of fuzzy measures not depending on its modality (regardless of the value of λ), clarity indicator will be equal 0. Consequently, at full uncertainty of measure distribution we obtain the minimum value of the index, it corresponds to the statement of the problem

Implementation of the third property (monotonicity property) of the proposed clarity index of fuzzy measures distribution can easily be verified experimentally. Let us consider the distribution of fuzzy measures densities g1 - g7 shown in (Fig. 3,a). For these distributions following condition is satisfied $\forall A \subseteq X,\ \ s,t = \overline{1,7},\ \ s > t,\ \ g_X^s(A) \geq g_X^t(A)$. For example, for the subset $A = \{x_5, x_6, x_7\}$ measures value $s = \overline{1,7},\ \ g_X^s(A)$ and values of clarity indicators $J(g_X^s)$ distribution of these fuzzy measures is presented in (Fig. 4).

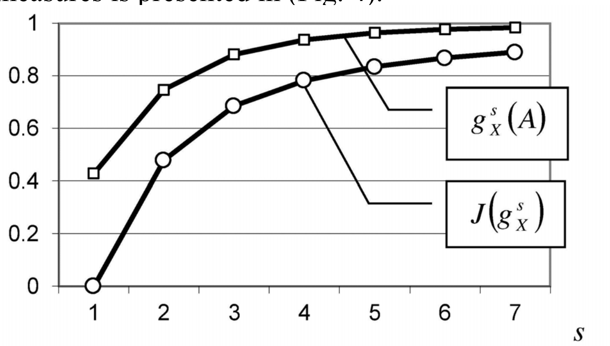


Fig. 4. Monotony property of clarity index of fuzzy measures distribution.

Thus, the proposed construction of the clarity indicator of fuzzy measure distribution satisfies the nominated requirements and can be used to determine the uncertainty level of the specified measures distribution.

It is important to note that the construction of the proposed clarity indicator implies the possibility of reordering density function of fuzzy measure $h(x)$ ascending so that the condition $\forall x, y \in X,\ x \leq y,\ \ h(x) \leq h(y) \in [0,1]$ satisfied, which allows to implement function integration $\delta(x) : X \to [-1,1]$ over the variable $x \in X$. In practice, this procedure can be difficult, especially in the analysis of measures distribution on the continuous space X. In order to avoid such a procedure in the existing clarity indices on a continuous space transition to α-level sets is used [3]. This approach allows to consider the increasing sequence of α-level subsets for the fuzzy set and go to integration by α level. This method can be directly used to construct the proposed clarity indicator only in the case where for the function $\delta(x)$ there is a functional relationship between $x \in X$ and $\alpha \in [0,1]$.

However, this condition is extremely rare. If the function of measures density has domains in which

$h(x)=\alpha$, then the direct application of transition to α-level sets is impossible due to the fact that the measure is a function of the set. Therefore, we consider a modification of the proposed clarity indicator by using for its representation the α-level approach of sets ordered sequence determination.

To derive the clarity indicator of fuzzy measures distribution $g_X(\cdot):2^X \to [0,1]$ we first consider the discrete space X.

Lemma 3. The clarity index for distribution of an arbitrary fuzzy measures $g_X(\cdot):2^X \to [0,1]$ on the discrete space X is determined by the following expression:

$$J(g_X)=\frac{1}{(I-1)\cdot\lambda}\cdot\left\{\sum_{j=\overline{1,N}}\left(a_{\alpha_j}\cdot c_{\alpha_j}-b_{\alpha_{j-1}}\right)\cdot\right.$$
$$\left.\cdot\left(c_{\alpha_1}^{N_{\alpha_j}-1}+c_{\alpha_1}^{N_{\alpha_j}-2}+\cdots+c_{\alpha_1}+1\right)-1\right\}.$$

where N - the number of α-levels density distribution of fuzzy measure $h(x_i)\in[0,1],\ i=\overline{1,I}$, $N=\bigcup_{\alpha_j} N_{\alpha_j}$. N_{α_j} - amount of points $x_i,\ i=\overline{1,I}$ of space X, for which $h(x_i)=\alpha_j$, $c_{\alpha_j}=(1+\lambda\cdot\alpha_j)$, $a_{\alpha_j}=\prod_{x_i\in G_j}(1+\lambda\cdot h(x_i))$, $b_{\alpha_{j-1}}=\prod_{x_i\in G_{j-1}}(1+\lambda\cdot h(x_i))$.

Proof. Let density distribution of measure is set as illustrated in (Fig. 5). For clarity of derivation, we will consider the function of measure density $h(x_i)\in[0,1],\ i=\overline{1,I}$ which has areas with the same measure density.

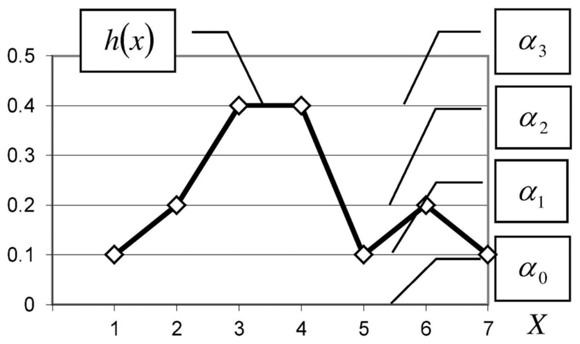


Fig. 5. Derivation of clarity indicator on discrete space.

Let's define a subsets family of strict α-level for a given fuzzy measure density as follows:

$$G_0=\left\{x_i\in X\,\middle|\,h(x_i)>\alpha_0,\ i=\overline{1,I}\right\},$$
$$G_1=\left\{x_i\in X\,\middle|\,h(x_i)>\alpha_1,\ i=\overline{1,I}\right\},\text{ etc.}$$

Then the construction of clarity measures can be represented as a step by step procedure:

$$\delta_0=g(G_0)-g(\overline{G}_0)=$$
$$=g(G_1\cup\{x_1,x_5,x_7\})-g(\overline{G}_0)=1$$
$$\delta_1=g(G_1\cup\{x_5,x_7\})-g(\overline{G}_0\cup\{x_1\}),$$
$$\delta_5=g(G_1\cup\{x_7\})-g(\overline{G}_0\cup\{x_1,x_5\})$$
$$\delta_7=g(G_1)-g(\overline{G}_0\cup\{x_1,x_5,x_7\})=$$
$$=g(G_2\cup\{x_2,x_6\})-g(\overline{G}_1),$$
$$\delta_2=g(G_2\cup\{x_6\})-g(\overline{G}_1\cup\{x_2\}),$$
$$\delta_6=g(G_2)-g(\overline{G}_1\cup\{x_2,x_6\})=$$
$$=g(G_3\cup\{x_3,x_4\})-g(\overline{G}_2),$$
$$\delta_3=g(G_3\cup\{x_4\})-g(\overline{G}_2\cup\{x_3\}),$$
$$\delta_4=g(G_3)-g(\overline{G}_2\cup\{x_3,x_4\})=$$
$$=g(G_4)-g(\overline{G}_3)=-1,$$

where $\overline{G}_0=G_4=\varnothing$, $g(\overline{G}_0)=g(G_4)=0$.

From this expansion clear that the values $\delta_i,\ i\in\{0,1,5\}$ associated with α-level α_1. Let us denote I_{α_1} a subset of the index of given values δ_i. Similarly, the values $\delta_i,\ i\in\{2,7\}$ associated with the level α_2, and $\delta_i,\ i\in\{3,6\}$ associated with the level α_3. Their indices subset will be denoted by analogy I_{α_2} and I_{α_3}, correspondingly. Then all indices set I can be presented as follows: $I=I_{\alpha_1}\cup I_{\alpha_2}\cup I_{\alpha_3}$.

Based on the expression (4), and introduced notations, the clarity index for a given fuzzy measure on a discrete space is defined as:

$$J(g_X)=\frac{1}{I-1}\cdot\sum_{i=\overline{0,I}}\delta_i=$$
$$=\frac{1}{I-1}\cdot\left[\sum_{i\in I_{\alpha_1}}\delta_i+\sum_{i\in I_{\alpha_2}}\delta_i+\sum_{i\in I_{\alpha_3}}\delta_i+\delta_4\right].$$

Let us consider term δ_1 in the given expression. These elements may be represented as:

$$g\left(G_1 \cup \{x_5, x_7\}\right) = = \frac{1}{\lambda}\left(\prod_{x_i \in G_1}\left(1+\lambda \cdot h(x_i)\right) \cdot \prod_{i=5,7}\left(1+\lambda \cdot h(x_i)\right) - 1\right).$$

$$g\left(\bar{G}_0 \cup \{x_1\}\right) = = \frac{1}{\lambda}\left(\prod_{x_i \in \bar{G}_0}\left(1+\lambda \cdot h(x_i)\right) \cdot \left(1+\lambda \cdot h(x_1)\right) - 1\right).$$

However, for the measure density we have $\forall i \in \{1,5,7\},\ h(x_i) = \alpha_1$. We introduce the notation: $(1+\lambda \cdot \alpha_1) = c_{\alpha_1}$, $\prod_{x_i \in G_1}(1+\lambda \cdot h(x_i)) = a_{\alpha_1}$ and $\prod_{x_i \in \bar{G}_0}(1+\lambda \cdot h(x_i)) = b_{\alpha_0}$. Then the expression for the term δ_1 takes the form:

$$\delta_1 = \frac{1}{\lambda}\left(a_{\alpha_1} \cdot c_{\alpha_1}^2 - b_{\alpha_0} \cdot c_{\alpha_1}\right) = \frac{c_{\alpha_1}}{\lambda}\left(a_{\alpha_1} \cdot c_{\alpha_1} - b_{\alpha_0}\right).$$

Using the notation introduced above we consider the sum of:

$$\sum_{i \in I_{\alpha_1}} \delta_i = \delta_0 + \delta_1 + \delta_5 = \frac{1}{\lambda}\left(a_{\alpha_1} \cdot c_{\alpha_1}^3 - b_{\alpha_0}\right) + + \frac{1}{\lambda}\left(a_{\alpha_1} \cdot c_{\alpha_1}^2 - b_{\alpha_0} \cdot c_{\alpha_1}\right) + \frac{1}{\lambda}\left(a_{\alpha_1} \cdot c_{\alpha_1} - b_{\alpha_0} \cdot c_{\alpha_1}^2\right) = = \frac{1}{\lambda}\left(a_{\alpha_1} \cdot c_{\alpha_1} - b_{\alpha_0}\right) \cdot \left(c_{\alpha_1}^2 + c_{\alpha_1} + 1\right).$$

Let's present remaining sums by analogy:

$$\sum_{i \in I_{\alpha_2}} \delta_i = \delta_3 + \delta_4 = \frac{1}{\lambda}\left(a_{\alpha_2} \cdot c_{\alpha_2}^2 - b_{\alpha_1}\right) + + \frac{1}{\lambda}\left(a_{\alpha_2} \cdot c_{\alpha_2} - b_{\alpha_1} \cdot c_{\alpha_2}\right) = = \frac{1}{\lambda}\left(a_{\alpha_2} \cdot c_{\alpha_2} - b_{\alpha_1}\right) \cdot \left(c_{\alpha_2} + 1\right)$$

$$\sum_{i \in I_{\alpha_3}} \delta_i = \delta_5 + \delta_6 = \frac{1}{\lambda}\left(a_{\alpha_3} \cdot c_{\alpha_3}^2 - b_{\alpha_2}\right) + + \frac{1}{\lambda}\left(a_{\alpha_3} \cdot c_{\alpha_3} - b_{\alpha_2} \cdot c_{\alpha_3}\right) = \frac{1}{\lambda}\left(a_{\alpha_3} \cdot c_{\alpha_3} - b_{\alpha_2}\right) \cdot \left(c_{\alpha_3} + 1\right).$$

On the basis of these relationships can be traced a stable pattern of representation of partial sums tied to α-level of fuzzy measure density distribution. Summarizing the results, we can write a general expression for the clarity index of fuzzy measures distribution:

$$J(g_X) = \frac{1}{I-1} \cdot \left[\sum_{i \in I_{\alpha_1}} \delta_i + \sum_{i \in I_{\alpha_2}} \delta_i + \sum_{i \in I_{\alpha_3}} \delta_i + \delta_4\right] = = \frac{1}{(I-1) \cdot \lambda} \cdot \left\{\sum_{j=\overline{1,N}}\left(a_{\alpha_j} \cdot c_{\alpha_j} - b_{\alpha_{j-1}}\right) \cdot \cdot \left(c_{\alpha_1}^{N_{\alpha_j}-1} + c_{\alpha_1}^{N_{\alpha_j}-2} + \cdots + c_{\alpha_1} + 1\right) - 1\right\}.$$

where N - number of α-levels distribution of fuzzy measure density, $N = \bigcup_{\alpha_j} N_{\alpha_j}$. N_{α_j} - number of points $x_i, i = \overline{1,I}$ of space X, for which $h(x_i) = \alpha_j$, $a_{\alpha_j} = \prod_{x_i \in G_j}(1+\lambda \cdot h(x_i))$, $b_{\alpha_{j-1}} = \prod_{x_i \in \bar{G}_{j-1}}\left(1+\lambda \cdot h(x_i)\right)$, $c_{\alpha_j} = \left(1+\lambda \cdot \alpha_j\right)$.

Let us consider constructing of clarity index of fuzzy measures distribution for the case of continuous space X. As previously noted for the continuous distribution is difficult to re-order measures density function by ascending. Transition to α-level subsets is difficult in the case where there is domains of definition of measures density function, for which the condition $h(x) = \alpha = const$ holds. For continuous space clarity index defined by the relationship (3).

Lemma 4. The clarity index for distribution of an arbitrary fuzzy measures $g_X(\cdot): 2^X \to [0,1]$ on continuous space X with density function $h(x): X \to [0,1]$ defined by the expression of the form:

$$J(g_X) = \frac{1}{Card(X)} \cdot \cdot \int_{\alpha \in [0,1]} \frac{e^{\lambda \cdot \alpha \cdot (c_\alpha - b_\alpha)} - 1}{\lambda^2 \cdot \alpha} \cdot \left\{p\left(G_{\alpha^+}\right) \cdot e^{-\lambda \cdot \alpha} - p\left(\bar{G}_\alpha\right)\right\} d\alpha. \quad (7)$$

where c_α, b_α the ends of the segment $[c_\alpha, b_\alpha]$, where the measures density function satisfies

$h(x)=\alpha\in[0,1], G_\alpha=\{x\in X|h(x)\ge\alpha\}$,

$G_{\alpha^+}=\{x\in X|h(x)>\alpha\}$, $p(\bar{G}_\alpha)=e^{\lambda\cdot\int_{\bar{G}_\alpha}h(x)dx}$,

$p(G_{\alpha^+})=e^{\lambda\cdot\int_{G_{\alpha^+}}h(x)dx}$.

Proof. To illustrate the proof we consider the distribution of fuzzy measure density $h(x): X\to[0,1]$, shown in (Fig. 6).

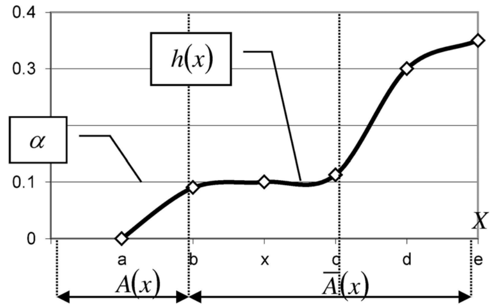


Fig. 6. The function of the fuzzy measures density.

The measure of subset $A\subseteq X$ on the continuous X is defined by the expression [9]:

$$g_X(A)=\frac{1}{\lambda}\cdot\left\{e^{\lambda\cdot\int_{A\subseteq X}h(x)dx}-1\right\}.$$

For the point $x\in X$ in the case of a non-decreasing measures density function $h(x)$ the value of the function $\delta(x)$, that is used in the clarity indicator $J(g_X)$ will be as follows:

$$\delta(x)=g_X\left(\bar{A}(x)\right)-g_X\left(A(x)\right)=$$
$$=\frac{1}{\lambda}\cdot\left\{e^{\lambda\cdot\int_{\bar{A}}h(x)dx}-e^{\lambda\cdot\int_{A}h(x)dx}\right\}.$$

The use of this function in the index $J(g_X)$ on the condition of an arbitrary distribution $h(x)$ is difficult. Therefore for ordering of subsets should be used α-level sets for a given distribution of the measure density. It will allows to make a change of the variable in the expression of clarity index and proceed to the integration over α. To derive the formula we consider the point $x\in[b,c]\subset X$, which lies on area with a constant density of the measure $h(x)=\alpha=const$ (Fig. 6). We introduce, for an arbitrary $\alpha\in[0,1]$ sets of normal $G_\alpha=\{x\in X|h(x)\ge\alpha\in[0,1]\}$ and strict $G_{\alpha^+}=\{x\in X|h(x)>\alpha\in[0,1]\}$ α-level.

Then, complement of α-level set will be $\bar{G}_\alpha=X\setminus G_\alpha$. Using this notation we can write: $A(x)=\bar{G}_\alpha\cup[b,x]$ and $\bar{A}(x)=G_{\alpha^+}\cup[x,c]$. Let us consider the second term in the brackets for values of the function $\delta(x)$:

$$e^{\lambda\cdot\int_{A}h(x)dx}=e^{\lambda\cdot\int_{\bar{G}_\alpha\cup[b,x]}h(x)dx}=p(\bar{G}_\alpha)\cdot e^{\lambda\cdot\int_{[b,x]}h(x)dx}.$$

Considering that on the interval $[b,x]$ the density measure $h(x)=\alpha$ we may write:

$$p(\bar{G}_\alpha)\cdot e^{\lambda\cdot\int_{[b,x]}h(x)dx}=p(\bar{G}_\alpha)\cdot e^{\lambda\cdot\alpha\cdot(x-b)}.$$

Similarly, we expand the first term in the brackets for values of the function $\delta(x)$:

$$e^{\lambda\cdot\int_{\bar{A}}h(x)dx}=e^{\lambda\cdot\int_{G_{\alpha^+}\cup[x,c]\setminus\{x\}}h(x)dx}=$$
$$=p(G_{\alpha^+})\cdot e^{\lambda\cdot\int_{[x,c]\setminus\{x\}}h(x)dx}=$$
$$=p(G_{\alpha^+})\cdot e^{\lambda\cdot\alpha\cdot(c-x)}\cdot e^{-\lambda\cdot\alpha}.$$

Then, the value of the function $\delta(x)$ will be:

$$\delta(x)=$$
$$=\frac{1}{\lambda}\left[p(G_{\alpha^+})\cdot e^{\lambda\cdot\alpha\cdot(c-x)}\cdot e^{-\lambda\cdot\alpha}-p(\bar{G}_\alpha)\cdot e^{\lambda\cdot\alpha\cdot(x-b)}\right]$$

In the absence of a function area with a constant density of the measure at the point $x\in X$ (functional relationship is performed $h(x)=\alpha,\ h^{-1}(\alpha)=x$) the following condition is satisfied:

$$c\to b\Rightarrow\begin{cases}\exp\left(\lambda\cdot\alpha\cdot(c-x)\right)\to1,\\ \exp\left(\lambda\cdot\alpha\cdot(x-b)\right)\to1.\end{cases}$$

Then the value of the function at the point $x\in X$ will be determined by expression:

$$\delta(x)=\frac{1}{\lambda}\left[p(G_{\alpha^+})\cdot e^{-\lambda\cdot\alpha}-p(\bar{G}_\alpha)\right].$$

This function will only depend on the value α and we can make the change of variable:

$\delta(x)=\delta(\alpha)$. However, in a case when there is an area $\forall x\in[b,c]\subseteq X,\, h(x)=\alpha=const$ the value $\delta(\alpha)$ will be determined by the integral:

$$\delta(\alpha)=\int\limits_{[b,c]}\delta(x)\,dx=$$

$$=\frac{p\left(G_{\alpha^+}\right)\cdot e^{-\lambda\cdot\alpha}}{\lambda}\cdot\int\limits_{[b,c]}e^{\lambda\cdot\alpha\cdot(c-x)}dx-$$

$$-\frac{p\left(\bar{G}_\alpha\right)}{\lambda}\cdot\int\limits_{[b,c]}e^{\lambda\cdot\alpha\cdot(x-b)}dx=$$

$$=\frac{\left(e^{\lambda\cdot\alpha\cdot(c-b)}-1\right)}{\lambda^2\cdot\alpha}\cdot\left\{p\left(G_{\alpha^+}\right)\cdot e^{-\lambda\cdot\alpha}-p\left(\bar{G}_\alpha\right)\right\}$$

Is evident that in the case of the functional relationship between α and x, we have a particular case of the expression obtained for the function $\delta(\alpha)$. Based on the foregoing expression for the function $\delta(\alpha)$ at the transition to the parameter α clarity index will be determined by the ratio:

$$J\left(g_X\right)=\frac{1}{Card\left(X\right)}\cdot\int\limits_{\alpha\in[0,1]}\delta(\alpha)\,d\alpha=\frac{1}{Card\left(X\right)}\cdot$$

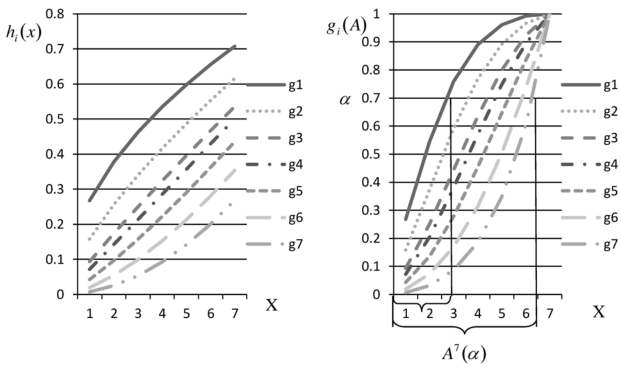
$$\cdot\int\limits_{\alpha\in[0,1]}\frac{\left(e^{\lambda\cdot\alpha\cdot(c_\alpha-b_\alpha)}-1\right)}{\lambda^2\cdot\alpha}\cdot\left\{p\left(G_{\alpha^+}\right)\cdot e^{-\lambda\cdot\alpha}-p\left(\bar{G}_\alpha\right)\right\}d\alpha$$

where c_α, b_α the ends of the segment $[c_\alpha, b_\alpha]$, where the measure density function satisfies the condition $h(x)=\alpha=const$. Q.E.D.

Thus, the transition to the parameter $\alpha\in[0,1]$ allows you to calculate clarity index of an arbitrary fuzzy measures distribution, both on discrete and continuous space X.

6. The discussion of the results

The proposed clarity index allows to consider the distribution modality of fuzzy measure density. Consider a family of fuzzy measures with different modalities $FM=\left\{g_X^i(\cdot)\,\middle|\, i=\overline{1,7}\right\}$, shown in (Fig.7,a)



a *b*

Fig. 7. The family of fuzzy measures. On a) the density of a measure, on b) the value of a measure.

Fuzzy measures of indicated family are distinguished by their densities. In this case following condition is satisfied: $h_i(x)=(h_0(x))^{\beta_i},\quad i=\overline{1,7}$, where $h_i(x)$ distribution density function of i-th measures, $\beta_i\geq 0,\quad i=\overline{1,7}$ - tension-compression factor for i-th measures, $h_0(x)$ basic linear function of the measures distribution density of the form: $h_0(x)=0.0714\cdot x$. The dependence of measures λ- parameter from tension-compression β_i is shown in (Fig. 8).

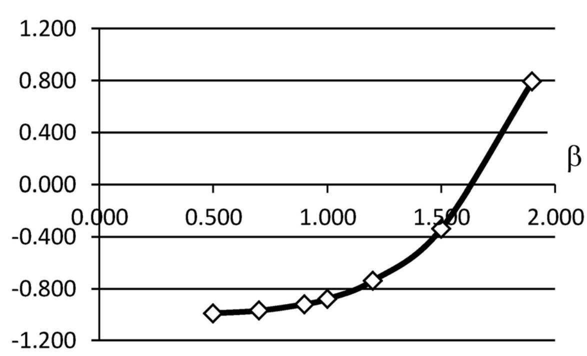


Fig. 8. Relationship between measures λ - parameter on tension-compression β_i for a family of measures.

Based on analysis of the given fuzzy measures family (Fig. 7,b), is seen that clarity index for fuzzy measures $g_X^7(\cdot): 2^X \to [0,1]$ must have a maximum value, as the concentration area of the measure more "Narrower" than in the other measures. What conclusion follows from the fact that to achieve the value of the measure for example $\alpha = 0.7$ (Fig. 7, b) interval $A^7(\alpha = 0.7) \subseteq X$ for measures $g_X^7(\cdot)$ should be much broader than that of other measure, therefore, the density of the fuzzy measures $h_7(x)$ with large values is concentrated in a smaller subset $\overline{A^7}(\alpha = 0.7) \subseteq X$ (Fig. 7,a). It should be noted that the measure $g_X^7(\cdot)$ is a measure closer to the necessity measure than other family measures as it has a maximum value of λ (Fig. 8).

The calculation results of clarity indicators for a given family of fuzzy measures are presented in (Fig. 9).

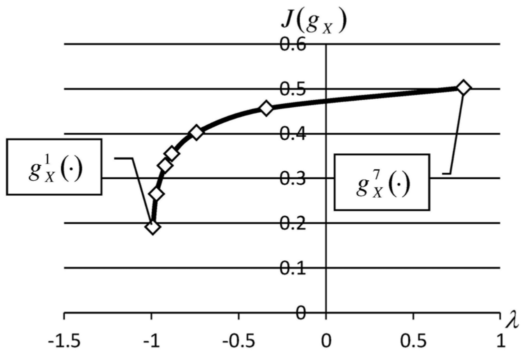


Fig. 9. Dependence of clarity indicator distribution of fuzzy measures for the selected family of measures on measures λ-parameter.

The graph shows that the clarity index of fuzzy measures distribution in the case of non-uniform 8density distribution is sensitive to changes in measures modality. The figure shows that our intuitive expectations that $J(g_X^7) > J(g_X^1)$ are confirmed by the results of the calculation.

7. Conclusion

Thus, we can say that the index clarity given in this article is an effective tool for analyzing the distribution of fuzzy measures and can be widely used for solving analytical problems under conditions of uncertainty with an arbitrary distribution modality of fuzziness degrees.

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Мета: обґрунтувати необхідність і запропонувати новий показник чіткості для розподілу нечіткої міри з довільної модальністю. Провести дослідження нового показника чіткості і показати ефективність, чутливість і простоту його використання для аналізу нечітких даних. **Методи:** для вирішення завдання використовуються методи теорії множин, теорії нечітких мір, а також функціональний аналіз і формальна логіка. Результати: обгрунтований і запропонований новий показник, який забезпечує отримання оцінки чіткості для розподілу нечіткої міри з довільної модальністю. Запропоновано формульні залежності для розрахунку показника чіткості нечіткої міри на дискретному і безперервному просторі. Доведено, що запропонований показник задовольняє властивостям, які висуваються до показників чіткості. Отримано додаткові залежності для розрахунку показника чіткості на основі використання множин рівня для функції щільності нечіткої міри. **Обговорення:** наведені результати розрахунку показника чіткості для сімейства нечітких мір з різними модальностями. Показано, що запропонований показник повністю задовольняє висунутим вимогам до логіки роботи показника чіткості і враховує модальність нечіткої міри.

Ключові слова: множини; нечітка міра; показник чіткості; якість управління

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Цель: обосновать необходимость и предложить новый показатель чёткости для распределения нечёткой меры с произвольной модальностью. Провести исследование нового показателя чёткости и показать эффективность, чувствительность и простоту его использования для анализа нечетких данных. **Методы:** для решения задачи используются методы теории множеств, теории нечетких мер, а также функциональный анализ и формальная логика. **Результаты:** обоснован и предложен новый показатель, который обеспечивает получение оценки чёткости для распределения нечёткой меры с произвольной модальностью. Предложены формульные зависимости для расчета показателя четкости нечеткой меры на дискретном и непрерывном пространстве. Доказано, что предложенный показатель удовлетворяет свойствам, которые выдвигаются к показателям четкости. Получены дополнительные зависимости для расчета показателя четкости на основе использования множеств уровня для функции плотности нечеткой меры. **Обсуждение**: приведены результаты расчета показателя чёткости для семейства нечётких мер с различными модальностями. Показано, что предложенный показатель полностью удовлетворяет выдвинутым требованиям к логике работы показателя чёткости и учитывает модальность нечёткой меры.

Ключевые слова: качество управления; множества; нечеткая мера; показатель четкости

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