ОБРОБКА СИГНАЛІВ

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GENERALIZATION OF GRONWALL-BELLMAN AND BEEKHARY LEMMAS

Well known lemmas touching useful integral inequalities for positive functions may be generalized. This paper contains some generalized lemmas similar to the Gronwall-Bellman and Beekhary lemmas.

The problem of paper is generalization of some known lemmas [1] delivering integral inequalities useful in dynamical system stability and controllability theory.

Lemma 1. Suppose that functions $u(t)\ge 0$; $f(t)\ge 0$; $g(t)\ge 0$ for all $t\ge t_0$ belong to the set $C[t_0,\infty)$, integral $\int_{t_0}^{t} g(s)f(s)ds$ exists and is finite. Then if inequality

$$u(t)\le g(t)+h(t)\int_{t_0}^{t} f(s)u(s)ds \tag{1}$$

is correct for function $u(t)$ next estimation has place:

$$u(t)\le g(t)+h(t)\int_{t_0}^{t} f(s)g(s)e^{\int_s^{\tau} f(\tau)h(\tau)d\tau}ds \tag{2}$$

for any $t\ge t_0$.

Proof. It can be found from (1) that

$$\frac{u(t)}{h(t)}\le\frac{g(t)}{h(t)}+\int_{t_0}^{t} f(s)u(s)ds. \tag{3}$$

Let as put

$$\frac{g(t)}{h(t)}=\beta(t)-\frac{\dot\beta(t)}{f(t)h(t)}, \tag{4}$$

where $\dot\beta(t)$ is derivative from $\beta(t)$ with respect to t, as it follows from inequality (3) and (4) next inequality has place:

$$\frac{u(t)f(t)+\dot\beta(t)}{\int_{t_0}^{t} f(s)u(s)ds+\beta(t)}\le f(t)h(t)$$

whence

$$\ln\left[\beta(t)+\int_{t_0}^{t} f(s)u(s)ds\right]_{t_0}^{t}\le\int_{t_0}^{t} f(s)h(s)ds. \tag{5}$$

It is seen from relationship (4) that

$$\beta(t)=e^{\int_{t_0}^{t} f(\tau)h(\tau)d\tau}\left[\beta_0-\int_{t_0}^{t} g(s)f(s)e^{\int_{t_0}^{s} f(\tau)h(\tau)d\tau}ds\right]. \quad (6)$$

Second term inside of square brackets is negative because according to the conditions all functions included in integrand expression are nonnegative, and integral $\int_{t_0}^{t} g(s)f(s)ds$ exists, is finite and

$$\int_{t_0}^{t} g(s)f(s)ds \ge \int_{t_0}^{t} g(s)f(s)e^{\int_{t_0}^{s} f(\tau)h(\tau)d\tau}ds.$$

Because constant $\hat{\beta}_0$ is arbitrary and taking into account that all functions in (6) are known such value $\beta_0 > 0$ can be choose that for fixed t_0 and t_1 function $\beta(t)$ will be positive. Then for all $t < t_1$ function $\beta(t)$ is funned to be positive too. For instance we can put

$$\beta_0=\int_{t_0}^{t_1} g(s)f(s)ds,\ t_1 > t.$$

Hence from (5) follows:

$$\int_{t_0}^{t} g(s)u(s)ds \le -\beta(t)+\beta_0 e^{\int_{t_0}^{t} f(s)h(s)ds}. \quad (7)$$

Substituting $\beta(t)$ according to (6) into above inequality we can receive relationship (2). We notice that final result does not depend on β_0 and consequentey (7) is true for any β_0.

Lemma 2. Suppose that $u(t)\ge 0$; $K(t,s)\ge 0$ in rectangle: $t\in[t_0,t]$; $s\in[t_0,t]$; $c>0$; $K(t,s)-$ non-increasing function of t and besides $u(t)$ and $K(t,s)\in C^{(1)}[0,\infty)$ and $K(t,s)$ continuously differentiable with respect to t. Then from inequality

$$u(t)\le c+\int_{t_0}^{t} K(t,s)u(s)ds \quad (8)$$

follows another inequality

$$u(t)\le Ce^{\int_{t_0}^{t} K(s,s)ds}.$$

Proof. We obtain from (8):

$$\frac{u(t)K(t,t)}{c+\int_{t_0}^{t} K(t,s)u(s)ds}\le K(t,t),$$

whence

$$\frac{u(t)K(t,t)+\int_{t_0}^{t}\frac{\partial K(t,s)}{\partial t}u(s)ds-\int_{t_0}^{t}\frac{\partial K(t,s)}{\partial t}u(s)ds}{c+\int_{t_0}^{t}K(t,s)u(s)ds}\le K(t,t).$$

Because $K(t,s)$ according to conditions is non-increasing with t function and consequently $\frac{\partial K(t,s)}{\partial t}\le 0$ then integral in numerator is negative and inequality became only sharpened if last term in numerator would be rejected. so:

$$\frac{u(t)K(t,t)+\int_{t_0}^{t}\frac{\partial K(t,s)}{\partial t}u(s)ds}{c+\int_{t_0}^{t}K(t,s)u(s)ds}\le K(t,t).$$

Integrating with t right side and left side of above inequality we would find

$$\ln\left[c+\int_{t_0}^{t}K(t,s)u(s)ds\right]\Bigg|_{t_0}^{t}\le\int_{t_0}^{t}K(s,s)ds.$$

Expression in square brackets is positive, $c>0$ therefore next inequality is true:

$$u(t)\le Ce^{\int_{t_0}^{t}K(s,s)ds}.$$

Let us consider some generalization of Beekhary lemma for nonlinear integral inequality.

Lemma 3. Suppose that functions $u(t)\ge 0$; $f(t)\ge 0$; $g(t)\ge 0$ and function $u(t)$ belong to the set $C[0,\infty)$; $t\ge t_0$.

Let next nonlinear inequality is true

$$u(t)\le g(t)+\int_{t_0}^{t}f_{(x)}\Phi(u(s))\,ds. \qquad (9)$$

Introduce function

$$\Psi(w)=\int_{w_0}^{w}\frac{dw}{\Phi(w)},$$

where $\Phi(w)$ is positive, continuous, non-decreasing function determined in the domain $w\in[0,w<\infty]$ and satisfies to the L ipshits condition

$$|\Phi(w_1)-\Phi(w_e)|\le L|W_1-W_2|, \qquad (10)$$

where $L\le 1$. Then if

$$\int_{t_0}^{t}f(s)ds<\Psi(\overline{W}_0)$$

for all such t that $t_0\le t<\infty$ next estimation has place

$$u(t) \le \Psi^{-1}\left[\int_{t_0}^{t} f(s)ds\right] + g(t) + \int_{t_0}^{t} g(s)f(s)e^{\int_s^t f(\tau)d\tau} ds. \quad (11)$$

Generalization comparatively with Beekhary lemma is only next one: instead of constant factor in right side of source inequality we put non negative but in other respects arbitrary function.

Proof. Consider in the same way as in case of proof of lemma 1 differential equation

$$\dot{\beta}(t) - f(t)\beta(t) = f(t)g(t). \quad (12)$$

Supposing arbitrary integrating constant β_0 to be equal zero: $\beta_0 = 0$, we can write

$$\beta(t) = \int_{t_0}^{t} g(s)f(s)e^{\int_s^t f(\tau)d\tau} ds. \quad (13)$$

It is seen that $\beta(t) \ge 0$ because all functions in integrand are non negative and $\beta = 0$ only if $g(t) \equiv 0$ for $t \ge t_0$. It could be found from (9) and (12), that

$$u(t) - \frac{\dot{\beta}(t)}{f(t)} \le -\beta(t) + \int_{t_0}^{t} f(s)\Phi(u(s))ds.$$

By virtue of non-decreasing nature of function $\Phi(w)$ we can obtain next inequality

$$\Phi\left[u(t) - \frac{\dot{\beta}(t)}{f(t)}\right] \le \Phi[-\beta(t) + \Phi(u(s))\,ds]. \quad (14)$$

Because $\Phi(w)$ satisfies Lipshits condition then substitution in (10) $(w_1 - w_1)$ instead of w_2 leads to the inequality

$$|\Phi(w_1) - \Phi(w_1 - w_2)| \le L|W_2|.$$

Setting $w_1 = u(t)$ and $w_2 = \dot{\beta}(t)f^{(-1)}(t)$ we can find:

$$\left|\Phi(u(t)) - \Phi\left(u(t) - \frac{\dot{\beta}(t)}{f(t)}\right)\right| \le L\left|\frac{\dot{\beta}(t)}{f(t)}\right|. \quad (15)$$

Since $u(t) \ge 0$ it is seen from (12) that $\dot{\beta}(t)f^{(-1)}(t) > 0$ and as false $\Phi(w)$ is non-decreasing function next inequality is true

$$\Phi(u(t)) - \Phi\left(u(t) - \frac{\dot{\beta}(t)}{f(t)}\right) > 0.$$

Because of that signs of absolute values in (15) can be taken away.

Then

$$\Phi(u(t)) - \Phi\left(u(t) - \frac{\dot{\beta}(t)}{f(t)}\right) \le L\frac{\dot{\beta}(t)}{f(t)} \le \frac{\dot{\beta}(t)}{f(t)},$$

where it was settled that $L = 1$. Taking into account (14) we can obtain:

$$\Phi(u(t)) - \frac{\dot{\beta}(t)}{f(t)} \le \Phi\left[-\beta(t) + \int_{t_0}^{t} f(s)\Phi(u(t))ds\right],$$

from which in its turn

$$\frac{\Phi(u(t))\cdot f(t)-\beta(t)}{\Phi\left[\int_{t_0}^{t} f(s)\Phi(u(s))ds-\beta(t)\right]}\le f(t).$$

Using designation $w=\int_{t_0}^{t} f(s)\Phi(u(s))ds-\beta(t)$, previous inequality to be presented in its turn

$$\frac{\dot{w}(t)dt}{\Phi(w)}\le f(t)dt.$$

Utter integrating in both sides next inequality should be found:

$$\Psi(w)=\int_{w_0}^{w}\frac{dw}{\Phi(w)}\le\int_{t_0}^{t} f(t)dt.$$

Because of condition $\beta(t_0)=0$ it is seen that $w_0=0$ and from the last relation follows

$$\frac{d\Psi(w)}{dw}=\frac{1}{\Phi(w)}>0$$

($\Phi(w)$ is positive when $0\le w\le\overline{w}<\infty$). Consequently monotone non-decreasing reciprocal function exists and designating $\Psi(w)=v$ we have right to write $w=\Psi^{-1}(v)$. Therefore from $v\le\int_{t_0}^{t} f(s)ds$ follows

$$w\le\Psi^{-1}\left[\int_{t_0}^{t} f(s)ds\right]. \tag{16}$$

But

$$w(t)\ge u(t)-\frac{\dot{\beta}(t)}{f(t)}=u(t)-\beta(t)-g(t),$$

or

$$u(t)\le w(t)+\beta(t)+g(t).$$

Substituting instead of $w(t)$ right hand side from (16) that make inequality more strong and replaying $\beta(t)$ in accordance with (13) we obtain inequality (11).

Lemma 4. Suppose the conditions of previous lemma are valid and additional supposition that function $K(t,s)$ is negative and non-increasing with respect to the first argument t, $t\in[t_0,t_1<\infty]$ and consequently has in this region non-positive partial derivative with respect t_0 is valid too. Then inequality

$$u(t)\le\Psi^{-1}\left[\int_{t_0}^{t} K(s,s)ds\right]+g(t)+\int_{t_0}^{t} g(s)K(s,s)e^{\int_{t_0}^{t}K(\tau,\tau)d\tau}ds$$

follows from inequality

$$u(t)\le g(t)+\int_{t_0}^{t} K(t,s)\Phi(u(s))ds.$$

Here as in previous lemma $\Psi^{-1}(v)$ is function reciprocal to the function

$$\bar{f}(w)=\bar{f}\left(\tau^*,\tau^*,w(s)\right)=\sup\ f(\tau,\tau,w(s)) \quad \tau\in[t_0,t_1<\infty].$$

$$u(t)\le\psi^{-1}\left[\int_0^w\frac{dw}{\bar{f}(w)}\right]+g(t)+L^*\int_{t_0}^t g(s)e^{L(t-s)}ds.$$

Proof. Let's put $g(t)=\dot{\varphi}(t)-L^*\varphi(t)$. Then if $\varphi_0=\varphi(t_0)=0$,

$$\varphi(t)=\int_{t_0}^t g(s)e^{L(t-s)}ds.$$

$$f(t,s,u(t)-\dot{\varphi}(t))\le f\left(t,s-L^*\varphi(s)+\int_{t_0}^s f(s,\tau,u(t)dt)\right).$$

$$\bar{f}\left(\tau^*,\tau^*,u(s)-\dot{\varphi}(s)\right)\le\bar{f}\left(\tau^*,\tau^*,-L^*\varphi(s)+\int_{t_0}^{\bar{s}} f(s,\tau,u(\tau))dt\right)$$

$$\left|\bar{f}(u(s))-\bar{f}(u(s)-\dot{\varphi}(s))\right|\le L^*\left|\dot{\varphi}(s)\right|.$$

$$\bar f(u(s)) - L^*\dot\varphi(s) \le \bar f(u(s) - \dot\varphi(s)) \le \bar f\left(-L^*\varphi(s) + \int_{t_0}^{s} \bar f(u(\tau)d\tau)\right).$$

It is seen that

$$\int_{t_0}^{t} \frac{\bar f(u(s)) - L^*\dot\varphi(s)}{\bar f\left(\int_{t_0}^{s} \bar f(u(t))d\tau - L^*\varphi(s)\right)} ds \le \int_{t_0}^{t} ds = t - t_0 .$$

Designating $w(s) = \int_{t_0}^{s} \bar f(u(\tau))d\tau - L^*\varphi(s)$ we can write

$$\psi(w) = \int_{w_0}^{w} \frac{dw}{\bar f(w)} \le t - t_0 = v,$$

because $\varphi(t_0) = \varphi_0 = 0$ then $w_0 = w(t_0) = 0$ and finally it could be written that

$$w(s) \ge u(s) - \dot\varphi(s) = u(s) - g(s) - L^*\varphi(s)$$

and

$$u(t) \le \psi^{-1}\left[\int_0^w \frac{dw}{\bar f(w)}\right] + g(t) + L^* \int_{t_0}^{t} g(s) e^{L^*(t-s)} ds .$$

Lemmas mentioned above could be used in particular for analysis of accessible and controllable sets of linear and nonlinear dynamical systems as well as for investigation of economical processes in models, which operates with essentially positive functions.

References

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