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PROJECTIVE CONNECTION SPACE

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The article deals with the differential geometry of (n+1)-dimensional affine space A_{n+1} . The three-component distributions ($H(M(\Lambda))$ -distributions) of affine space are discussed. The geometrical objects $\{h^p\}$, $\{\lambda^p\}$, which are quasitensors of the second order of the three-component distribution, have been constructed. These objects determine the normals of the first kind of the basic distribution by an inner invariant method in the second differential neighborhood of forming element of the three-component distribution. It has been proved that projective connection Γ for the three-component distribution is defined by an inner way in the differential neighborhood of the second order and belongs to the type of the projective connections which are defined by the way of projection.

The geometry of the distributions of multidimensional spaces is intensively studied from various reasons. Necessity of such research is caused by numerous generalizations and direct applications to physics, mechanics, the theory of the optimization, the calculus of variations.

Prof. A.V. Stolyarov, Y.I. Popov and M.M. Pohila have studied the methods of generalizations of the theory of distributions in multidimensional affine, projective spaces and in spaces of projective connections.

Prof. M.M. Pohila articles deal with the differential geometry of n-dimensional zones in affine and projective spaces [1].

Prof. A.V. Stolyarov introduces and discusses the Geometry of regular hyperzone distribution in multidimensional projective space [2]. He defined four affine connections with the torsion and curvature, gave geometrical interpretation of the connections that had been obtained.

Prof. Y.I. Popov demonstrated application of the multi-component distributions to the theory of regular and vanishing hyperzones and hyperzone distributions in projective spaces [3].

The aim of the article is studying projective connections of three-component distributions allowing generalization of the theory of regular and vanishing hyperzone, zone, hyperzone distribution, surfaces of full and non-full rang, tangentially equipped surfaces in multidimensional spaces.

The results of research can be applied to general theory of distributions in multidimensional spaces and to the theory of projective connections, which are associated with the three-component distribu-

tions. We use results, which we have got in the articles [4-6]. Throughout all summary the indexes take the following meanings:

1. Let us consider (n+1)-dimensional affine space A_{n+1} , which is taken to a movable frame

$$R = \{A, \bar{e}_I\}.$$

Differential equations of the infinitesimal transference of frame R look as follows:

$$dA = \omega^I \bar{e}_I;$$

$$d\bar{e}_I = \omega_I^K \bar{e}_K,$$

where ω_I^K, ω^I – invariant forms of an affine group, which satisfy the equations of the structure:

$$d\omega^I = \omega^K \wedge \omega_K^I;$$

$$d\omega_I^K = \omega_I^J \wedge \omega_J^K.$$

Three of distributions of affine space A_{n+1} , consisting of basic distribution of the first kind r-dimensional linear elements $\Pi_r = \Lambda$ (Λ -distribution), equipping distribution of the first kind m-dimensional linear elements $\Pi_m = M$ (M -distribution) and equipping distribution of the first kind of hyperplane elements $\Pi_n = H$ (H -distribution) with relation to the incidence of their corresponding elements in the common center A of the following view: $A \in \Pi_r \subset \Pi_m \subset \Pi_n$ are called $H(M(\Lambda))$ -distribution.

We will place vectors \bar{e}_p in the plane Π_r , vectors \bar{e}_m – in the plane Π_m , and vectors \bar{e}_n – in the plane Π_n . Such frame will be called frame of the null order R^0 .

Partial canonization of zero-order frame R^0 is possible, where

$$M_{iq}^{n+1} = 0, H_{\alpha q}^{n+1} = 0.$$

We will call it frame of the first order R^1 .

2. Let us construct the determinants

$$a_0 = \det \|a_{pq}\|; \tag{1}$$

$$d = \det \|a_{ij}\|;$$

where

$$a_{pq} = \frac{1}{2}(\Lambda_{pq} + \Lambda_{qp});$$

$$\nabla a_{pq} + a_{pq} \omega_{n+1}^{n+1} = a_{pqK} \omega^K;$$

$$a_{ij} = \frac{1}{2}(M_{ij} + M_{ji});$$

$$\nabla a_{ij} + a_{ij} \omega_{n+1}^{n+1} = a_{ijK} \omega^K.$$

In the general case the tensors $\{a_{pq}\}$, $\{a_{ij}\}$ are nondegenerated, so it is possible to bring in the tensors $\{p^{pq}\}$ and $\{a^{kj}\}$.

By differentiating the equations (1) we have:

$$d \ln a_0 = -r \omega_{n+1}^{n+1} + 2 \omega_p^p + a_K \omega^K;$$

$$d \ln d = 2 \omega_k^k - (m-r) \omega_{n+1}^{n+1} + a^{ji} a_{ijk} \omega^K,$$

where

$$a_K = a^{qt} a_{tqK};$$

$$d_K = a^{ji} a_{ijk}.$$

Then we will obtain the following quasitensors of the second order:

$$\lambda^p = -\frac{1}{r+2} \Lambda^{qp} a_q^0;$$

$$h^p = -\frac{1}{m-r} \Lambda^{qp} d_q,$$

where

$$\nabla \lambda^p - \lambda^p \omega_{n+1}^{n+1} + \omega_{n+1}^p = \lambda_K^p \omega^K;$$

$$\nabla h^p - h^p \omega_{n+1}^{n+1} + \omega_{n+1}^p = h_K^p \omega^K.$$

The geometrical objects $\{h^p\}$, $\{\lambda^p\}$ determine the inner invariant normals of the first kind for the basic Λ -distribution in the differential neighborhood of the second order of the three-component distribution forming element.

3. Let us consider the space of the projective connection $P_{n+1,r}$, which has been determined in the following way: affine space A_{n+1} is $(n+1)$ -dimensional base of this space and the r -dimensional planes Π_r of the basic Λ -distribution (r -dimensional centered affine space) are layers of this space.

The projective connection Γ of space $P_{n+1,r}$ is always determined by the system of the forms [7; 8]:

$$\theta^p = \omega^p - \Gamma_{oK}^p \omega^K;$$

$$\theta_q^p = \omega_q^p - \Gamma_{qK}^p \omega^K.$$

The transformed forms θ^p, θ_q^p satisfy the following structural equations:

$$D\theta^p = \theta^q \wedge \theta_q^p + \omega^K \wedge \Delta \Gamma_{oK}^p;$$

$$D\theta_q^p = \theta_q^r \wedge \theta_r^p + \omega^K \wedge \Delta \Gamma_{qK}^p,$$

where

$$\Delta \Gamma_{oK}^p = \nabla \Gamma_{oK}^p + \delta_K^{n+1} \omega_{n+1}^p - \Gamma_{qK}^p \omega^q - \Gamma_{oK}^q \Gamma_{qJ}^p \omega^J - A_{uK}^p \omega^u;$$

$$\Delta \Gamma_{qK}^p = \nabla \Gamma_{qK}^p + \Lambda_{qK} \omega_{n+1}^p + (\Lambda_{qK}^u A_{uJ}^p - \Gamma_{qK}^r \Gamma_{rJ}^p) \omega^J.$$

The forms $\Delta \Gamma_{oK}^p, \Delta \Gamma_{qK}^p, \omega^K$ on the $H(M(\Lambda))$ -distribution constitute the completely integrable system and determine the field of the geometrical object $\{\Gamma_{oK}^p, \Gamma_{qK}^p\}$ over the initial base (ω_o^K) . The geometrical object $\{\Gamma_{oK}^p, \Gamma_{qK}^p\}$ will be called the object of the projective connection [8] of the space $P_{n+1,r}$.

To determine projective connection in the layers (in the planes Π_r) of the space $P_{n+1,r}$ by the forms θ^p, θ_q^p , it is necessary and sufficient to determine the field of the object of the connection $\{\Gamma_{oK}^p, \Gamma_{qK}^p\}$, i.e. for the differential equations to be fulfilled [7]:

$$\Delta \Gamma_{oK}^p = \Gamma_{oK}^p \omega^L;$$

$$\Delta \Gamma_{qK}^p = \Gamma_{qK}^p \omega^L. \tag{2}$$

Structural equations for layer forms θ^p, θ_q^p of the space $P_{n+1,r}$ look as follows:

$$D\theta^p = \theta^q \wedge \theta_q^p + \frac{1}{2} R_{oKL}^p \omega^K \wedge \omega^L;$$

$$D\theta_q^p = \theta_q^r \wedge \theta_r^p + \frac{1}{2} R_{qKL}^p \omega^K \wedge \omega^L, \tag{3}$$

where $\{R_{oKL}^p, R_{qKL}^p\}$ is the torsion-curvature tensor of the projective connection Γ of the space $P_{n+1,r}$ and we have:

$$R_{oKL}^p = 2\Gamma_{o[KL]}^p;$$

$$R_{qKL}^p = 2\Gamma_{q[KL]}^p.$$

4. We will construct the projective connection Γ , which is defined by the three-component $H(M(\Lambda))$ -distribution by an inner way, i.e. we will construct the scope of the object of the projective connection $\Gamma = \{\Gamma_{oK}^p, \Gamma_{qK}^p\}$ by the fundamental objects of the $H(M(\Lambda))$ -distribution.

Let us represent the system of the differential equations (2) in the following form:

$$\nabla \Gamma_{oq}^p = \tilde{\Gamma}_{oqK}^p \omega^K;$$

$$\nabla \Gamma_{oi}^p = \tilde{\Gamma}_{oiK}^p \omega^K;$$

$$\nabla \Gamma_{\alpha\alpha}^p = \tilde{\Gamma}_{\alpha\alpha K}^p \omega^K; \tag{4}$$

$$\nabla \Gamma_{on+1}^p + \omega_{n+1}^p - \Gamma_{on+1}^p \omega_{n+1}^{n+1} - \Gamma_{oq}^p \omega_{n+1}^q - \Gamma_{oi}^p \omega_{n+1}^i = \tilde{\Gamma}_{on+1K}^p \omega^K;$$

$$\begin{aligned} \nabla \Gamma_{qs}^p + \Lambda_{qs} \omega_{n+1}^p &= \tilde{\Gamma}_{qsK}^p \omega^K; \\ \nabla \Gamma_{qi}^p + \Lambda_{qi} \omega_{n+1}^p &= \tilde{\Gamma}_{qiK}^p \omega^K; \\ \nabla \Gamma_{q\alpha}^p + \Lambda_{q\alpha} \omega_{n+1}^p &= \tilde{\Gamma}_{q\alpha K}^p \omega^K; \\ \nabla \Gamma_{qn+1}^p - \Gamma_{qj}^p \omega_{n+1}^j + \Lambda_{qn+1} \omega_{n+1}^p &= \tilde{\Gamma}_{qn+1K}^p \omega^K. \end{aligned} \quad (5)$$

It is easy to check that the equations (4), (5) will be satisfied if the scopes of the components of the object of the projective connection $\{\Gamma_{\alpha K}^p, \Gamma_{qK}^p\}$ are realized in the following way:

$$\begin{aligned} \Gamma_{oq}^p &= 0; \Gamma_{ou}^p = -b_u^{pq} \ell_q; \Gamma_{on+1}^p = a^u b_u^{pq} \ell_q + v^p; \\ \Gamma_{qK}^p &= \Lambda_{qK} (a^u b_u^{pq} \ell_r + v^p) - \Lambda_{qK}^u b_u^{pr} \ell_r. \end{aligned} \quad (6)$$

Thus it is proved that the forms θ^p, θ_q^p , which define the layer of the space of the projective connection $P_{n+1,r}$, determined on the $H(M(\Lambda))$ -distribution by an inner way and is associated with the basic Λ -distribution, look as follows:

$$\begin{aligned} \theta^p &= \omega^p + b_u^{pq} \ell_q \omega^u - (a^u b_u^{pq} \ell_q + v^p) \omega^{n+1}; \\ \theta_q^p &= \omega_q^p - (\Lambda_{qK} a^u b_u^{pq} \ell_q + \Lambda_{qK} v^p - \Lambda_{qK}^u b_u^{pr} \ell_r) \omega^K, \end{aligned}$$

where the objects $\{h^p\}$ and $\{\lambda^p\}$, which were constructed in the part 2 of this article, can be taken as the object $\{v^p\}$.

5. According to the article [8] let us prove that the constructed projective connection Γ belongs to the type of the projective connections which are defined by the way of projection. Really, the plane

$$[\vec{A}(u, du), \vec{e}_p(u, du)] = \Pi_r(u, du)$$

is the image of the current plane

$$\Pi_r(u + du) \equiv [\vec{A}(u + du), \vec{e}_p(u + du)]$$

of the basic Λ -distribution in the case of the mapping

$$\begin{aligned} \vec{A}(u + du) &\rightarrow \vec{A}(u, du) = \vec{A}(u) + \omega^j \vec{e}_j(u) + [2]; \\ \vec{e}_p(u + du) &\rightarrow \vec{e}_p(u, du) = \vec{e}_p(u) + \omega_p^j \vec{e}_j(u) + [2], \end{aligned} \quad (7)$$

which determines the connection.

Let us project the image

$$[\vec{A}(u, du), \vec{e}_p(u, du)] = \Pi_r(u, du),$$

of neighboring plane

$$\Pi_r(u + du) \equiv [\vec{A}(u + du), \vec{e}_p(u + du)],$$

onto the current plane of the Λ -distribution

$$\Pi_r(u) \equiv [\vec{A}(u), \vec{e}_p(u)],$$

taking the equipping plane $K_{n+1}(A)$ as the projecting center.

This projection determines the reflection:

$$\vec{A}(u, du) \rightarrow \tilde{\vec{A}}(u, du) = \vec{A}(u, du) +$$

$$\begin{aligned} &+ \ell^{n+1} \vec{K}_{n+1} + \ell^u \vec{K}_u = \\ &= \vec{A}(u) + \omega^j \vec{e}_j + \ell^{n+1} (\vec{e}_{n+1} + H_{n+1}^p \vec{e}_p) + \\ &+ \ell^u (\vec{e}_u + H_u^p \vec{e}_p) = \\ &= \vec{A}(u) + (\omega^p + \ell^{n+1} H_{n+1}^p + \ell^u H_u^p) \vec{e}_p + \\ &+ (\omega^u + \ell^u) \vec{e}_u + (\omega^{n+1} + \ell^{n+1}) \vec{e}_{n+1}; \\ \vec{e}_p(u, du) &\rightarrow \tilde{\vec{e}}_p(u, du) = \vec{e}_p(u, du) + \ell_p^{n+1} \vec{K}_{n+1} + \\ &+ \ell_p^v \vec{K}_v = \vec{e}_p + \omega_p^j \vec{e}_j + \ell_p^{n+1} (\vec{e}_{n+1} + H_{n+1}^q \vec{e}_q) + \\ &+ \ell_p^v (\vec{e}_v + H_v^q \vec{e}_q) = \vec{e}_p(u) + \\ &+ (\omega_p^q + \ell_p^{n+1} H_{n+1}^q + \ell_p^v H_v^q) \vec{e}_q + \\ &+ (\omega_p^u + \ell_p^u) \vec{e}_u + (\omega_p^{n+1} + \ell_p^{n+1}) \vec{e}_{n+1}. \end{aligned} \quad (8)$$

We will determine the coefficients $\ell^{n+1}, \ell^u, \ell_p^{n+1}, \ell_p^v$ from the condition: the projections $\tilde{\vec{A}}(u, du), \tilde{\vec{e}}_p(u, du)$ of the vectors $\vec{A}(u, du), \vec{e}_p(u, du)$ must belong to the plane

$$\Pi_r(u) \equiv [\vec{A}(u), \vec{e}_p(u)],$$

i.e. the members with \vec{e}_u and \vec{e}_{n+1} must not be present in the equations (8).

As a result we get that:

$$\begin{aligned} \ell^u &= -\omega^u; \ell^{n+1} = -\omega^{n+1}; \\ \ell_p^u &= -\omega_p^u; \ell_p^{n+1} = -\omega_p^{n+1}. \end{aligned}$$

Thus, reflections superposition (7) and (8) gives the reflection determining projective connection on the $H(M(\Lambda))$ -distribution, received by method of projecting:

$$\begin{aligned} \vec{A}(u, du) &\rightarrow \tilde{\vec{A}}(u, du) = \vec{A}(u) + \theta^p \vec{e}_p; \\ \vec{e}_p(u, du) &\rightarrow \tilde{\vec{e}}_p(u, du) = \vec{e}_p(u) + \theta_q^p \vec{e}_q. \end{aligned}$$

Here the forms θ^p, θ_q^p :

$$\begin{aligned} \theta^p &= \omega^p - H_{n+1}^p \omega^{n+1} - H_u^p \omega^u; \\ \theta_q^p &= \omega_q^p - H_{n+1}^p \omega_q^{n+1} - H_v^p \omega_q^v \end{aligned}$$

determine the main part of the received reflection and are the forms of projective connection Γ on the $H(M(\Lambda))$ -distribution, which was determined by the projecting method.

The object of this connection is determined by the formulas (6).

6. The components of the torsion-curvature tensor of the space $P_{n+1,r}$ in the structural equations (3) look as follows:

$$\begin{aligned} R_{qKJ}^p &= 2(\Lambda_{q[KJ]} (a^u b_u^{pq} \ell_q + v^p) + \Lambda_{q[K} a_{J]}^u b_u^{pq} \ell_q + \\ &+ \Lambda_{q[K} b_{u]J}^{pq} a^u \ell_q + \Lambda_{q[K} \ell_{|q]J} a^u b_u^{pJ} + \\ &+ \Lambda_{q[K} v_{J]}^p - \Lambda_{q[KJ]}^u b_u^{pr} \ell_r - \Lambda_{q[K}^u b_{u]J}^{pr} \ell_r - \\ &- \Lambda_{q[K}^u \ell_{|r]J} b_u^{pr} - \Lambda_{q[K} \Lambda_{|r]J} a^u b_u^{rI} \ell_I + a^v b_v^{ps} \ell_s - \end{aligned}$$

$$\begin{aligned}
& -\Lambda_{q[K}\Lambda_{r|J]}v^r a^v b_v^{ps} \ell_t + \Lambda_{q[K}\Lambda_{r|J]}b_u^{rt} \ell_t a^v b_v^{ps} \ell_s - \\
& -\Lambda_{q[K}\Lambda_{r|J]}a^u b_u^{rt} \ell_t v^p - \Lambda_{q[K}\Lambda_{r|J]}v^r v^p + \\
& + \Lambda_{q[K}\Lambda_{r|J]}b_u^{rt} \ell_t v^p + \Lambda_{q[K}\Lambda_{r|J]}a^u b_u^{rt} \ell_t b_v^{ps} \ell_s + \\
& + \Lambda_{q[K}\Lambda_{r|J]}v^r b_v^{ps} \ell_s - \Lambda_{q[K}\Lambda_{r|J]}b_u^{rt} \ell_t b_v^{ps} \ell_s \Big); \\
R_{oKL}^p &= 2\Gamma_{o[KL]}^p,
\end{aligned}$$

where

$$\begin{aligned}
\Gamma_{ouj}^p &= -b_{uj}^{pq} \ell_q - b_u^{pq} \ell_{qj} - \delta_j^q \Lambda_{qu} a^v b_v^{pt} \ell_t - \\
& - \delta_j^q \Lambda_{qu} v^p + \delta_j^q \Lambda_{qu} b_v^{pr} \ell_r + \Lambda_{qj} b_u^{qr} a^v \ell_t b_v^{pt} \ell_t + \\
& + v^p \ell_r b_u^{qr} \Lambda_{qj} - b_u^{qr} \Lambda_{qj} \ell_r b_v^{pt} \ell_t - A_{ju}^p \delta_j^u; \\
\Gamma_{oql}^p &= -\delta_l^r \Lambda_{rq} a^u b_u^{pt} \ell_t - \delta_l^r \Lambda_{rq} v^p + \\
& + \delta_l^r \Lambda_{rq} b_u^{pt} \ell_t - A_{uq}^p \delta_l^u; \\
\Gamma_{on+1j}^p &= a_j^u b_u^{pq} \ell_q + v_j^p + a^u b_{uj}^{pq} \ell_q + a^u b_u^{pq} \ell_{qj} + \\
& + a^u b_u^{pq} \ell_{qj} - \delta_j^q (\Lambda_{qn+1} a^u b_u^{pt} \ell_t + v^p \Lambda_{qn+1} - \Lambda_{qn+1}^u b_u^{pr} \ell_r) - \\
& - a^u b_u^{qr} \ell_r \Lambda_{qj} a^v b_v^{pt} \ell_t - \delta_j^q A_{un+1}^p - v^q \Lambda_{qj} a^u b_u^{pt} \ell_t - \\
& - a^u b_u^{qr} \ell_r \Lambda_{qj} v^p - v^q \Lambda_{qj} v^p + \\
& + a^u b_u^{qr} \ell_r \Lambda_{qj} b_v^{pt} \ell_t + v^q \Lambda_{qj} b_u^{pt} \ell_t.
\end{aligned}$$

The torsion-curvature tensor determines the projective connection space.

The geometrical objects $\{h^p\}$, $\{\lambda^p\}$, which are quasitensors of the second order of the three-component distribution, have been constructed.

These objects determine the normals of the first kind of the basic distribution by the inner invariant method in the second differential neighborhood of forming element of the three-component distribution. It is proved that projective connection Γ for the three-component distribution is defined by an inner way in the differential neighborhood of the second order and belongs to the type of the projective connections which are defined by the way of the projection.

The results of research can be applied to general theory of distributions in multidimensional spaces and to theory of projective connections, which are associated with the three-component distributions.

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Простір проективної зв'язності

Розглянуто трискладові розподіли (Н(М(Λ))-розподіли) афінного простору. Доведено, що проективна зв'язність Γ для трискладового розподілу визначена внутрішнім чином у диференціальному оточенні другого порядку і належить до типу проективних зв'язностей. Отримано компоненти тензора кривини – скруту простору проективної зв'язності.

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Пространство проективной связности

Рассмотрены трехсоставные распределения (Н(М(Λ))-распределения) афинного пространства. Доказано, что проективная связность Γ для трехсоставного распределения определена внутренним образом в дифференциальной окрестности второго порядка и принадлежит к типу проективных связностей. Получены компоненты тензора кривизны – кручения пространства проективной связности.

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