AEROSPACE SYSTEMS OF MONITORING AND MANAGEMENT

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MODELING OF AIRCRAFTS COLLISION USING IMPORTANCE SAMPLING TECHNIQUE

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We introduce the model of movement of two conflicting aircrafts and state the problem, we apply the importance sampling technique and elaborate an algorithm of collision modeling based on normal distributions, a small simulation study.

Introduction

In this paper we use the importance sampling technique in order to model rare events of aircrafts collision.

Model

Suppose that an aircraft is flying along its aerial corridor. We are interested in its current coordinate x (this could be either an altitude or transverse coordinate). We assume that the x coordinate of the aircraft has random distribution with probability density (p.d.) of mixing type,

$$f(x;\mu)=(1-\alpha)\frac{1}{2a_1b_1\Gamma(b_1)}\exp\left(-\left|\frac{x-\mu}{a_1}\right|^{1/b_1}\right)+$$

$$+\alpha\frac{1}{2a_2b_2\Gamma(b_2)}\exp\left(-\left|\frac{x-\mu}{a_2}\right|^{1/b_2}\right), x\in\Re. \qquad (1)$$

Here Gamma function $\Gamma(t)=\int_0^\infty x^{t-1}e^{-x}dx, t>0$, mixing parameter $\alpha\in(0,1)$, a_1 and a_2 are positive scale parameters, b_1 and b_2 are positive shape parameters, and $\mu\in\Re$ is location parameter corresponding to the axis of the aerial corridor. The p.d. of the form (1) is called density of double generalized Laplace (DGL) distribution. It is used to model a random walk of an aircraft in Reich's model. The first summand on the right hand side of (1) is a core of the distribution, while the second summand corresponds to rare positions where the aircraft occurs far from the axis of the corridor.

We consider two conflicting aircrafts flying in the neighboring corridors. The corresponding probability densities are $f(x_1;\mu_1)$ and $f(x_2;\mu_2)$, with different location parameters μ_1 and μ_2 and common mixing, scale and shape parameters α, a_1, a_2, b_1 and b_2. Let X_j be a current x-coordinate of j-th aircraft, $j=1,2$, and X be a compound vector (x_1,x_2) of the coordinates. We suppose that the two aircrafts are wandering independently in their corridors, therefore a joint p.d. of X is given by

$$f_X(x_1,x_2)=f(x_1;\mu_1)f(x_2;\mu_2),\ x_1,x_2\in\Re. \qquad (2)$$

The collision along the x-coordinate happens if $|X_1-X_2|\le d$, where d is an effective diameter of an aircraft. The problem is to estimate the probability P_x of collision along the x-coordinate,

$$P_x=P\{|X_1-X_2|\le d\}=\mathbf{E}I(|X_1-X_2|\le d).$$

Hereafter bold $\mathbf{E}$ denotes mathematical expectation, and $I(A)$ is indicator function of event A, i.e., $I(A)=1$ if A occurs while $I(A)=0$ if A does not occur. The collision is a typical rare event. Therefore straightforward Monte Carlo simulation is inefficient [1, p. 350–351]. Instead we apply importance sampling technique.

Modeling based on importance sampling

Instead of sampling $(X_1^{(1)},X_2^{(1)})$, $(X_1^{(2)},X_2^{(2)}),\ldots,$ $(X_1^{(n)},X_2^{(n)})$ from the distribution (2) and using

$$\sum_{i=1}^n I\left(\left|X_1^{(i)}-X_2^{(i)}\right|\le d\right)\Big/n,$$

we will sample $(X_1^{(1)},X_2^{(1)}),\ldots,(X_1^{(n)},X_2^{(n)})$ from a different distribution $\hat{f}_X(x_1,x_2)$ and use a statistic

$$Y_n=\frac{1}{n}\sum_{i=1}^n\frac{I\left(\left|X_1^{(i)}-X_2^{(i)}\right|\le d\right)\cdot f_X(X_1^{(i)},X_2^{(i)})}{\hat{f}_X(X_1^{(i)},X_2^{(i)})}. \qquad (3)$$

Note that $\mathbf{E}Y_n=\mathbf{E}I(|X_1-X_2|\le d)=P_x$, where (X_1,X_2) has distribution (2). By the Strong Law of

Large Numbers, $Y_n \to P_x$ with probability 1, as $n \to \infty$. Now, the problem is to choose the p.d. f_X satisfying two conditions: a) modeling of the event $|X_1^{(i)} - X_2^{(i)}| \le d$ should be realistic; b) the conversance $Y_n \to P_X$ should be as quick as possible.

We set $h(x) = h(x_1, x_2) = I(|x_1 - x_2| \le d)$.

The optimal choice of f_X is $\hat{f}_X(x) = f_X(x)h(x)/\mathbf{E}h(X)$, where $x = (x_1, x_2) \in \Re^2$ and $X = (X_1, X_2)$ has distribution (2), but this is impossible as $\mathbf{E}h(x)$ is unknown. We propose to choose

$$\hat{f}_X(x) = \frac{g_X(x)h(x)}{\iint\limits_{\Re^2} g_X(x)h(x)dx}, \qquad (4)$$

where $g_X(x_1, x_2) = g_{X_1}(x_1)g_{X2}(x_2)$, and $g_{X_j}(x_j)$ is a Gaussian p.d. having common expectation and variance with the DGL p.d. $f(x_j; \mu_j); j = 1, 2$. We will see that the usage of normal distributions makes our modeling realistic. And the choose of the parameters of the Gaussian distributions causes similarity of the distributions f_x and g_x, therefore the density (4) will be close to the optimal p.d. $f_X(x)h(x)/\mathbf{E}h(X)$, and the convergence $Y_n \to P_X$ should not be slow.

Now, we describe the modeling procedure. First we choose parameters of the normal lows. For $a, b > 0$, denote

$$\rho(x; a, b) = \frac{1}{2ab\Gamma(b)} \exp\left(-\left|\frac{x-\mu}{a}\right|^{1/b}\right), x \in \Re. \qquad (5)$$

Then $f(x;\mu) = (1-\alpha)\rho(x; a_1, b_1) + \alpha\rho(x; a_2, b_2)$. The distribution (5) is symmetric around μ, therefore its expectation equals μ. The variance $\sigma^2(a,b)$ of (5) is calculated easily, $\sigma^2(a,b) = \int\limits_{\Re} (x-\mu)^2 \rho(x; a, b)dx = \frac{a^2\Gamma(3b)}{\Gamma(b)}$.

Then the expectation of the distribution (1) is μ, and the variance is

$$\sigma^2 = (1-\alpha)\sigma^2(a_1, b_1) + \alpha\sigma^2(a_2, b_2) = (1-\alpha)\frac{a_1^2\Gamma(3b_1)}{\Gamma(b_1)} + \alpha\frac{a_2^2\Gamma(3b_2)}{\Gamma(b_2)}. \qquad (6)$$

Thus we choose $g_{X_j}(x_j)$ to be a p.d. of the distribution $N(\mu_j, \sigma^2)$, where σ^2 is given in (6); $j = 1, 2$.

Now, we organize sampling from the distribution (4) based on a normal random generator. Let $\hat{X} = (\hat{X}_1; \hat{X}_2)$ be a random vector with distribution $N((\mu_1, \mu_2), \sigma^2 I)$, where I is identity matrix. Then $\hat{X}_1 = \mu_1 + \sigma\hat{\gamma}_1$, and $\hat{X}_2 = \mu_2 + \sigma\hat{\gamma}_2$, where $\hat{\gamma}_1$, and $\hat{\gamma}_2$ are independent standard normal random variables. We have

$$\hat{X}_1 - \hat{X}_2 = \mu_1 - \mu_2 + \sigma\sqrt{2}\left(\frac{\hat{\gamma}_1 - \hat{\gamma}_2}{\sqrt{2}}\right).$$

Introduce random variables $\gamma_1 = \frac{\hat{\gamma}_1 - \hat{\gamma}_2}{\sqrt{2}}$ and $\gamma_2 = \frac{\hat{\gamma}_1 + \hat{\gamma}_2}{\sqrt{2}}$. Then γ_1 and γ_2 are independent standard normal variables as well. The event $|\hat{X}_1 - \hat{X}_2| \le d$ holds if and only if

$$\gamma_1 \in \left[\frac{\mu_1 - \mu_2 - d}{\sigma\sqrt{2}}, \frac{\mu_1 - \mu_2 + d}{\sigma\sqrt{2}}\right] = J.$$

Denominator in (4) equals

$$P\{|\hat{X}_1 - \hat{X}_2| \le d\} = P\{\gamma_1 \in J\} = \Phi\left(\frac{\mu_1 - \mu_2 + d}{\sigma\sqrt{2}}\right) - \Phi\left(\frac{\mu_1 - \mu_2 - d}{\sigma\sqrt{2}}\right),$$

where Φ – distribution function of standard normal law.

Thus p.d. (4) equals

$$\hat{f}_X(x_1, x_2) = \frac{g_{X_1}(x_1)g_{X_2}(x_2)I(|x_1 - x_2| \le d)}{\Phi\left(\frac{\mu_1 - \mu_2 + d}{\sigma\sqrt{2}}\right) - \Phi\left(\frac{\mu_1 - \mu_2 - d}{\sigma\sqrt{2}}\right)}. \qquad (7)$$

We simulate a sample from the p.d. (7) as follows. A standard normal generator produces $\gamma_2^{(1)}$, and then the generator works till we obtain the value $\gamma_1^{(1)} \in J$. We set

$$\hat{X}_1^{(1)} = \mu_1 + \frac{\sigma}{\sqrt{2}}\left(\gamma_1^{(1)} + \gamma_2^{(1)}\right); \hat{X}_2^{(1)} = \mu_2 + \frac{\sigma}{\sqrt{2}}\left(\gamma_2^{(1)} - \gamma_1^{(1)}\right).$$

Then the generator produces $\gamma_2^{(2)}$, and it works till we obtain $\gamma_1^{(2)} \in J$. We set

$$\hat{X}_1^{(2)} = \mu_1 + \frac{\sigma}{\sqrt{2}}\left(\gamma_1^{(2)} + \gamma_2^{(2)}\right);$$

$$\hat{X}_2^{(2)} = \mu_2 + \frac{\sigma}{\sqrt{2}}\left(\gamma_2^{(2)} - \gamma_1^{(2)}\right),$$

etc. For the simulated sample, the inequality $|\hat{X}_1^{(i)} - \hat{X}_2^{(i)}| \le d$ holds, for each i = 1, 2, …, n. According to (3), we set

$$Y_n = \frac{1}{n}\sum_{i=1}^{n} \frac{f_X(\hat{X}_1^{(i)}, \hat{X}_2^{(i)})}{\hat{f}_X(\hat{X}_1^{(i)}, \hat{X}_2^{(i)})},$$

with $\hat{f}_X$ given in (7).

As was mentioned above, Y_n is unbiased and strongly consistent estimator of P_x, i.e., $\mathbf{E}Y_n = P_x$ and $Y_n \to P_x$ a.s., as $n \to \infty$. The rate of convergence can be controlled by the variance

$$V[Y_n] = \frac{1}{n}\left(\iint_{\Re^2} \frac{I(|x_1 - x_2| \le d) f_X^2(x_1,x_2)}{\hat{f}_X(x_1,x_2)} dx_1 dx_2 - P_x \right).$$

Unfortunately, as usual in (1) b_2 > ½ holds, which means that the second summand in (1) has the tails of the distribution, which are heavier then the tails of normal law. This causes that for our choice of $\hat{f}_X$, the variance of Y_n is infinite. Nevertheless still $Y_n \to P_x$ a.s., as $n \to \infty$, and we illustrate the rate of convergence by the simulation study in Section 4.

Though the tails of normal law are typically lighter then the tails of basic distribution (1), we have strong arguments in favour of our choice of the p.d. (8). Suppose that we chose another p.d.

$$\tilde{f}_X(x_1,x_2) = \frac{\tilde{g}_X(x_1,x_2) I(|X_1 - X_2| \le d)}{\int_{\{(x_1,x_2): |x_1 - x_2| \le d\}} \tilde{g}_X(x_1,x_2) dx_1 dx_2}.$$

Let $\tilde{x} = (\tilde{x}_1, \tilde{x}_2)$ be a random vector with p.d. $\tilde{g}_X(x_1,x_2)$. In the simulation procedure, we want to reduce the dimension as was done above, and we want to simulate separately $\tilde{x}_1 - \tilde{x}_2$ and other linear combination $\alpha\tilde{x}_1 + \beta\tilde{x}_2$, with $\det\begin{pmatrix} 1 & -1 \\ \alpha & \beta \end{pmatrix} \ne 0$. Thus for independent random variables $\tilde{x}_1$ and $\tilde{x}_2$, we want that another couple $\tilde{x}_1 - \tilde{x}_2$ and $\alpha\tilde{x}_1 + \beta\tilde{x}_2$ be independent. But due to the theorem of characterization of normal law [2, p. 97], this implies that $\tilde{x}_1$ and $\tilde{x}_2$ are Gaussian, and we come back to the density (7).

Simulations

In this section results of calculations on the above-stated method are presented.

In fig. 1 dependence of theoretical probability of the accident on sample size designed on algorithm is shown with two underlying normal lows:

$$\sigma_1 = a_1; \sigma_2 = 225 a_2;$$

$$A_1 = \frac{\mu_2 - \mu_1 - d}{\sqrt{\sigma_1^2 + \sigma_2^2}}; A_2 = \frac{\mu_2 - \mu_1 + d}{\sqrt{\sigma_1^2 + \sigma_2^2}};$$

$$P_n = \Phi(A_2) - \Phi(A_1); \; P_n = 3{,}5289 \times 10^{-8}.$$

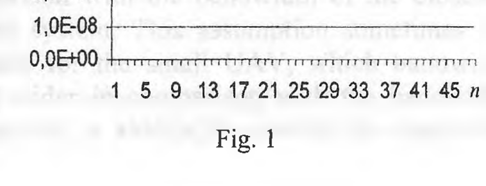

Fig. 1

In tab. 1 numerical values of experimental probability of accident are given, with mixing parameter $\alpha = 0$, which corresponds to normal densities.

Table 1

Experimental probability of accident (α = 0)

n	$Y_n \times 10^{-8}$
3	3,40930
9	2,53771
15	4,13542
21	3,47838
27	2,62164
33	3,37126
39	3,49381
45	3,28352
50	2,94175

In fig. 2 dependence of experimental probability of accident on sample size is shown.

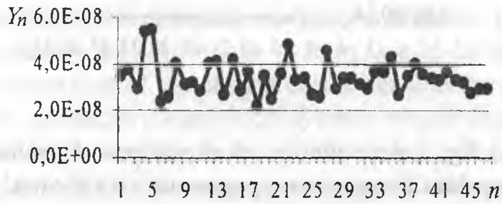

Fig. 2

In tab. 2 the numerical values of a relative error $|P_n - Y_n|/P_n$ are given.

Table 2

Relative error

n	$\lvert P_n - Y_n\rvert/P_n$
3	0,03389
9	0,28088
15	0,17187
21	0,01432
27	0,25709
33	0,11820
39	0,00994
45	0,06953
50	0,05977

In fig. 3 dependence of a relative error on sample size is shown, with sample size equal 50.

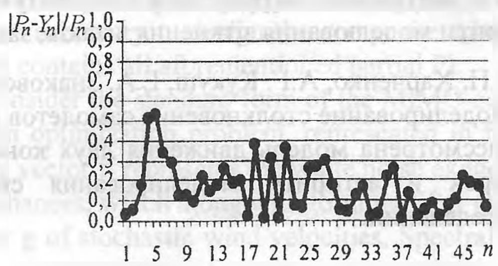

Fig. 3

In tab. 3 numerical values of experimental probability of accident are given, with mixing parameter $\alpha = 0{,}0136$.

Table 3

Experimental probability of accident (α=0,0136)

n	Y_n, $\times 10^{-8}$
3	1,04831
9	1,29658
15	1,68247
21	1,38736
27	1,20631
33	1,44253
39	1,20936
45	1,4257
50	1,37045

In fig. 4 dependence of experimental probability of accident on sample size is shown.

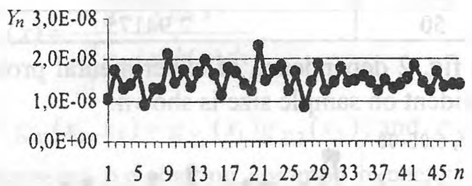

Fig. 4

In fig. 5 dependence of experimental probability of accident from mixing parameter α is shown.

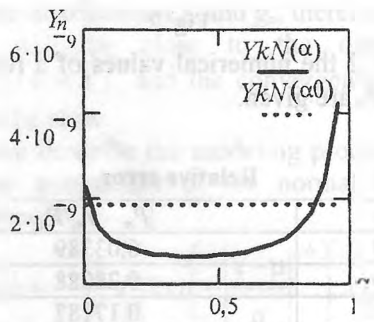

Fig. 5

In tab. 4 numerical values of experimental probability of accident are given.

Table 4

Experimental probability

α	YkN, $\times 10^{-9}$
0	2.273835796
0.1	1.022599989
0.2	0.7572907593
0.3	0.6583314607
0.4	0.6320845977
0.5	0.6571705710
0.6	0.7390390539
0.7	0.9129333553
0.8	1.294254661
0.9	2.401420192
0.95	2.273835796

Conclusions

We proposed an algorithm of modeling of aircrafts collision. The idea is to use importance sampling technique and sample from a different distribution, based on a normal law. This makes it possible to reduce the dimension of the problem and instead of a rare event for the two-dimensional random vector have e rare event for a one-dimensional random variable.

Simulation study shows that the proposed technique leads to reasonable results for commonly used probability densities. In the next research we will try to tune the parameters of basic distribution in order to obtain the prescribed probability of collision.

References

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Моделювання зіткнення літаків з використанням методики істотних вибірок

Розглянуто модель руху двох конфліктуючих літаків. Запропоновано методику істотних вибірок і алгоритм моделювання зіткнення літаків, заснований на нормальному розподілі.

В.П. Харченко, А.Г. Кукуш, Е.А. Знаковская

Моделирование столкновения самолетов при использовании методики важных выборок

Рассмотрена модель движения двух конфликтующих самолетов. Предложены методика важных выборок и алгоритм моделирования столкновения самолетов, основанный на нормальном распределении.