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# THE DENSITY DISTRIBUTION FUNCTION RENEWAL PROCESS OF THREE VARIABLES BY MEANS OF THE POLYNOMIAL SPLINE BASED ON THE B-SPLINES

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*The question of asymptotic property research of local polynomial spline of three variables on the basis of the second order B-splines is considered in the article. Both the correspondent theorem and arguments as well as implications were formulated, proved and mentioned.*

Let us consider some observation results of simultaneous realization according to three characteristics $T$, $Q$, $G$ of the observed and investigated object which are the arguments of some $p(t,q,g)$ function. Let us state three separations $\Delta_{h_t}$, $\Delta_{h_q}$, $\Delta_{h_g}$ of the axes observed by means of points $t_i = ih_t$, $q_j = jh_q$, $g_r = rh_r$, $(i,j,r \in Z)$, provided that the steps $h_t$, $h_q$, $h_g$ correspondingly to which the separations $\Delta_{h_t,h_q,h_g}$ of the simultaneous realization of $T$, $Q$, $G$ properties domain are given. As the result of the separations $\Delta_{h_t,h_q,h_g}$, there is $a\times b\times c$ of domains in the boundaries of which we shall look for the approximation function $p(t,q,g)$ according to the array of $\left\{(t_i,q_j,g_r),p_{i,j,r};i=\overline{1,a},j=\overline{1,b},r=\overline{1,c}\right\}$, where $(t_i,q_j,g_r)$ the interior point $(i,j,r)$ of domain separation; and $p_{i,j,r}$ – value of function $p(t_i,q_j,g_r)$.

According to [1], let us state the local polynomial spline on the basis of the second order B-splines in accordance with the array of values $P=\{p_{i,j,r};(i,j,r\in Z)\}$

$$S_{2,0}(p,t,q,g)= \tag{1}$$

$$=\sum_{i\in Z}\sum_{j\in Z}\sum_{r\in Z}B_{2,h_t}(t-ih_t)B_{2,h_q}(q-jh_q)B_{2,h_g}(g-rh_g)p_{i,j,r},$$

where with accuracy to the argument [2]

$$B_{2,h_l}(l)=\begin{cases} 0, & |l|\ge 3h_l/l, \\ (3+2l/h_l)^2/8, & l\in[-3h_l/2,-h_l/2], \\ 3/4-(2l/h_l)^2/4, & l\in[-h_l/2,h_l/2], \\ (3-2l/h_l)^2/8, & l\in[h_l/2,3h_l/2], \end{cases}$$

$l=t,q,g$.

For the value of function $p(t,q,g)$ on the $(i,j,r)$ separation element $p_{i,j,r}=\overline{p}_{i,j,r}+\varepsilon_{i,j,r}$, where

$$\overline{p}_{i,j,r}=\frac{1}{h_t h_q h_g}\int\limits_{(i-1/2)h_t}^{(i+1/2)h_t}\int\limits_{(j-1/2)h_q}^{(j+1/2)h_q}\int\limits_{(r-1/2)h_g}^{(r+1/2)h_g}p(t,q,g)\,dtdqdg,$$

$i,j,r\in Z$; $\varepsilon_{i,j,r}$ – a mistake;

we can write

$$|p(t,q,g)-S_{2,0}(p,t,q,g)|\le$$

$$\le|p(t,q,g)-S_{2,0}(\overline{p},t,q,g)|+\varepsilon\|S_{2,0}(p,t,q,g)\|, \tag{2}$$

where $\varepsilon=\max\limits_{i,j,r}\{\varepsilon_{i,j,r}\}$;

$$\|S_{2,0}(p,t,q,g)\|=\sup_{|\varepsilon_{i,j,r}|\le 1}\max_{t,q,g}|S_{2,0}(\varepsilon,t,q,g)|.$$

So, to determine the quality of approximation function $p(t,q,g)$ by means of spline (1), there is a problem how to evaluate each of the summands of the right part (2).

**Theorem 1.** If $p(t,q,g)\in C^{2,2,2}$ so $h_t\to 0$, $h_q\to 0$, $h_g\to 0$ and where $h=\max\{h_t,h_q,h_g\}$ then the asymptotic equality is fulfilled uniformly according to the arguments $t$, $q$, $g$

$$p(t,q,g)-S_{2,0}(\overline{p},t,q,g)=-\frac{h_t^2}{6}p''_{t^2}(t,q,g)-$$
$$-\frac{h_q^2}{6}p''_{q^2}(t,q,g)-\frac{h_g^2}{6}p''_{g^2}(t,q,g)-\frac{h_t^2h_q^2}{36}p^{(4)}_{t^2q^2}(t,q,g)-$$
$$-\frac{h_t^2h_g^2}{36}p^{(4)}_{t^2g^2}(t,q,g)-\frac{h_q^2h_g^2}{36}p^{(4)}_{q^2g^2}(t,q,g)-$$
$$-\frac{h_t^2h_q^2h_g^2}{216}p^{(6)}_{t^2q^2g^2}(t,q,g)+O(h^6). \tag{3}$$

**Proof.** Under Taylor-series expansion of the function $p(t,q,g)\in C^{2,2,2}$ near the points $(ih_t,jh_q,rh_g)$, we shall get

(let us write this in such a way $\tilde{t}=ih_t$, $\tau=t-\tilde{t}$; $\tilde{q}=jh_q$, $\nu=q-\tilde{q}$; $\tilde{g}=rh_g$, $\zeta=g-g*$, $p=p(ih_t,jh_q,jh_g)$, $p'_t=p'_t(ih_t,jh_q,rh_g)$, ... $S(\tilde{p},\tilde{t},\tilde{q},\tilde{g})=S(\tilde{p})$ ...):

$$p(t,q,g)=p+p'_t\tau+p'_q\nu+p'_q\zeta+\frac{1}{2}p''_{t^2}\tau^2+\frac{1}{2}p''_{q^2}\nu^2+$$

$$+\frac{1}{2}p''_{g^2}\zeta^2+p''_{tq}\tau\nu+p''_{tg}\tau\zeta+p''_{qg}\nu\zeta+\frac{1}{2}p'''_{t^2q}\tau^2\nu+$$
$$+\frac{1}{2}p'''_{tq^2}\tau\nu^2+\frac{1}{2}p'''_{t^2g}\tau^2\zeta+\frac{1}{2}p'''_{tg^2}\tau\zeta^2+\frac{1}{2}p'''_{q^2g}\nu^2\zeta+\frac{1}{2}p'''_{qg^2}\nu\zeta^2+$$
$$+p'''_{tqg}\tau\nu\zeta+\frac{1}{4}p^{(4)}_{t^2q^2}\tau^2\nu^2+\frac{1}{4}p^{(4)}_{t^2g^2}\tau^2\zeta^2+\frac{1}{4}p^{(4)}_{q^2g^2}\nu^2\zeta^2+$$
$$+\frac{1}{2}p^{(4)}_{t^2qg}\tau^2\nu\zeta+\frac{1}{2}p^{(4)}_{tq^2g}\tau\nu^2\zeta+\frac{1}{2}p^{(4)}_{tqg^2}\tau\nu\zeta^2+\frac{1}{4}p^{(5)}_{t^2q^2g}\tau^2\nu^2\zeta+$$
$$+\frac{1}{4}p^{(5)}_{t^2qg^2}\tau^2\nu\zeta^2+\frac{1}{4}p^{(5)}_{tq^2g^2}\tau\nu^2\zeta^2+\frac{1}{8}p^{(6)}_{t^2q^2g^2}\tau^2\nu^2\zeta^2+O\left(h^6\right).$$

For the spline $S_{2,0}\left(\bar{p},t,q,g\right)$, it will be correct respectively:

$$S_{2,0}(p,t,q,g)=S_{2,0}(\tilde{p})+S'_{2,0}(\tilde{p})_t\,\tau+S'_{2,0}(\tilde{p})_q\,\nu+$$
$$+S'_{2,0}(\tilde{p})_g\,\zeta+\frac{1}{2}S''_{2,0}(\tilde{p})_{t^2}\,\tau^2+\frac{1}{2}S''_{2,0}(\tilde{p})_{q^2}\,\nu^2+$$
$$+\frac{1}{2}S''_{2,0}(\tilde{p})_{g^2}\,\zeta^2+S''_{2,0}(\tilde{p})_{tq}\,\tau\nu+S''_{2,0}(\tilde{p})_{tg}\,\tau\zeta+$$
$$+S''_{2,0}(\tilde{p})_{qg}\,\nu\zeta+\frac{1}{2}S'''_{2,0}(\tilde{p})_{t^2q}\,\tau^2\nu+\frac{1}{2}S'''_{2,0}(\tilde{p})_{tq^2}\,\tau\nu^2+$$
$$+\frac{1}{2}S'''_{2,0}(\tilde{p})_{t^2g}\,\tau^2\zeta+\frac{1}{2}S'''_{2,0}(\tilde{p})_{tg^2}\,\tau\zeta^2+\frac{1}{2}S'''_{2,0}(\tilde{p})_{q^2g}\,\nu^2\zeta+$$
$$+\frac{1}{2}S'''_{2,0}(\tilde{p})_{qg^2}\,\nu\zeta^2+S'''_{2,0}(\tilde{p})_{tqg}\,\tau\nu\zeta+\frac{1}{4}S^{(4)}_{2,0}(\tilde{p})_{t^2q^2}\,\tau^2\nu^2+$$
$$+\frac{1}{4}S^{(4)}_{2,0}(\tilde{p})_{t^2g^2}\,\tau^2\zeta^2+\frac{1}{4}S^{(4)}_{2,0}(\tilde{p})_{q^2g^2}\,\nu^2\zeta^2+\frac{1}{2}S^{(4)}_{2,0}(\tilde{p})_{t^2qg}\,\tau^2\nu\zeta+$$
$$+\frac{1}{2}S^{(4)}_{2,0}(\tilde{p})_{tqg^2}\,\tau\nu\zeta^2+\frac{1}{4}S^{(5)}_{2,0}(\tilde{p})_{t^2q^2g}\,\tau^2\nu^2\zeta+$$
$$+\frac{1}{4}S^{(5)}_{2,0}(\tilde{p})_{t^2qg^2}\,\tau^2\nu\zeta^2+\frac{1}{4}S^{(5)}_{2,0}(\tilde{p})_{tq^2g^2}\,\tau\nu^2\zeta^2+$$
$$+\frac{1}{8}S^{(6)}_{2,0}(\tilde{p})_{t^2q^2g^2}\,\tau^2\nu^2\zeta^2+O\left(h^6\right),$$

where (because of economy motive, the part of expressions isn't given any more)

$$S_{2,0}(\tilde{p})=\frac{1}{512}\Big(\bar{p}_{i-1,j-1,r-1}+6\bar{p}_{i-1,j-1,r}+\bar{p}_{i-1,j-1,r+1}+$$
$$+6\bar{p}_{i-1,j,r-1}+36\bar{p}_{i-1,j,r}+6\bar{p}_{i-1,j,r+1}+\bar{p}_{i-1,j+1,r-1}+$$
$$+6\bar{p}_{i-1,j+1,r}+\bar{p}_{i-1,j+1,r+1}+6\bar{p}_{i,j-1,r-1}+36\bar{p}_{i,j-1,r}+6\bar{p}_{i,j-1,r+1}+$$
$$+36\bar{p}_{i,j,r-1}+216\bar{p}_{i,j,r}+36\bar{p}_{i,j,r+1}+6\bar{p}_{i,j+1,r-1}+36\bar{p}_{i,j+1,r}+$$
$$+6\bar{p}_{i,j+1,r+1}+\bar{p}_{i+1,j-1,r-1}+6\bar{p}_{i+1,j-1,r}+\bar{p}_{i+1,j-1,r+1}+$$
$$+6\bar{p}_{i+1,j,r-1}+36\bar{p}_{i+1,j,r}+6\bar{p}_{i+1,j,r+1}+\bar{p}_{i+1,j+1,r-1}+$$
$$+6\bar{p}_{i+1,j+1,r}+\bar{p}_{i+1,j+1,r+1}\Big);$$

$$S'_{2,0}(\tilde{p})_t=\frac{1}{128h_t}\Big(-\bar{p}_{i-1,j-1,r-1}-6\bar{p}_{i-1,j-1,r}-$$
$$-\bar{p}_{i-1,j-1,r+1}-6\bar{p}_{i-1,j,r-1}-36\bar{p}_{i-1,j,r}-6\bar{p}_{i-1,j,r+1}-\bar{p}_{i-1,j+1,r-1}-$$
$$-6\bar{p}_{i-1,j+1,r}-\bar{p}_{i-1,j+1,r+1}+\bar{p}_{i+1,j-1,r-1}+6\bar{p}_{i+1,j-1,r}+$$
$$+\bar{p}_{i+1,j-1,r+1}+6\bar{p}_{i+1,j,r-1}+36\bar{p}_{i+1,j,r}+6\bar{p}_{i+1,j,r+1}+$$
$$+\bar{p}_{i+1,j+1,r-1}+6\bar{p}_{i+1,j+1,r}+\bar{p}_{i+1,j+1,r+1}\Big);$$

$$S'_{2,0}(\tilde{p})_q=\frac{1}{128h_q}\Big(-\bar{p}_{i-1,j-1,r-1}-6\bar{p}_{i-1,j-1,r}-\bar{p}_{i-1,j-1,r+1}+$$
$$+\bar{p}_{i-1,j+1,r-1}+6\bar{p}_{i-1,j+1,r}+\bar{p}_{i-1,j+1,r+1}-6\bar{p}_{i,j-1,r-1}-$$
$$-36\bar{p}_{i,j-1,r}-6\bar{p}_{i,j-1,r+1}+6\bar{p}_{i,j+1,r-1}+36\bar{p}_{i,j+1,r}+6\bar{p}_{i,j+1,r+1}-$$
$$-\bar{p}_{i+1,j-1,r-1}-6\bar{p}_{i+1,j-1,r}-\bar{p}_{i+1,j-1,r+1}+$$
$$+\bar{p}_{i+1,j+1,r-1}+6\bar{p}_{i+1,j+1,r}+\bar{p}_{i+1,j+1,r+1}\Big);$$

$$S'_{2,0}(\tilde{p})_g=\frac{1}{128h_g}\Big(-\bar{p}_{i-1,j-1,r-1}+\bar{p}_{i-1,j-1,r+1}-$$
$$-6\bar{p}_{i-1,j,r-1}+6\bar{p}_{i-1,j,r+1}-\bar{p}_{i-1,j+1,r-1}+\bar{p}_{i-1,j+1,r+1}-$$
$$-6\bar{p}_{i,j-1,r-1}+6\bar{p}_{i,j-1,r+1}-36\bar{p}_{i,j,r-1}+36\bar{p}_{i,j,r+1}-6\bar{p}_{i,j+1,r-1}+$$
$$+6\bar{p}_{i,j+1,r+1}-\bar{p}_{i+1,j-1,r-1}+\bar{p}_{i+1,j-1,r+1}-6\bar{p}_{i+1,j,r-1}+$$
$$+6\bar{p}_{i+1,j,r+1}-\bar{p}_{i+1,j+1,r-1}+\bar{p}_{i+1,j+1,r+1}\Big);$$

$$S''_{2,0}(\tilde{p})_{t^2}=\frac{1}{64h_t^2}\Big(\bar{p}_{i-1,j-1,r-1}+6\bar{p}_{i-1,j-1,r}+$$
$$+\bar{p}_{i-1,j-1,r+1}+6\bar{p}_{i-1,j,r-1}+36\bar{p}_{i-1,j,r}+6\bar{p}_{i-1,j,r+1}+$$
$$+\bar{p}_{i-1,j+1,r-1}+6\bar{p}_{i-1,j+1,r}+\bar{p}_{i-1,j+1,r+1}-2\bar{p}_{i,j-1,r-1}-$$
$$-12\bar{p}_{i,j-1,r}-2\bar{p}_{i,j-1,r+1}-12\bar{p}_{i,j,r-1}-72\bar{p}_{i,j,r}-$$
$$-12\bar{p}_{i,j,r+1}-2\bar{p}_{i,j+1,r-1}-12\bar{p}_{i,j+1,r}-2\bar{p}_{i,j+1,r+1}+$$
$$+\bar{p}_{i+1,j-1,r-1}+6\bar{p}_{i+1,j-1,r}+\bar{p}_{i+1,j-1,r+1}+6\bar{p}_{i+1,j,r-1}+$$
$$+36\bar{p}_{i+1,j,r}+6\bar{p}_{i+1,j,r+1}+\bar{p}_{i+1,j+1,r-1}+6\bar{p}_{i+1,j+1,r}+\bar{p}_{i+1,j+1,r+1}\Big);$$

$$S''_{2,0}(\tilde{p})_{tq}=\frac{1}{32h_th_q}\Big(\bar{p}_{i-1,j-1,r-1}+6\bar{p}_{i-1,j-1,r}+$$
$$+\bar{p}_{i-1,j-1,r+1}-\bar{p}_{i-1,j+1,r-1}-6\bar{p}_{i-1,j+1,r}-$$
$$-\bar{p}_{i-1,j+1,r+1}-\bar{p}_{i+1,j-1,r-1}-6\bar{p}_{i+1,j-1,r}-\bar{p}_{i+1,j-1,r+1}+$$
$$+\bar{p}_{i+1,j+1,r-1}+6\bar{p}_{i+1,j+1,r}+\bar{p}_{i+1,j+1,r+1}\Big);$$

…

$$S^{(5)}_{2,0}(\tilde{p})_{t^2qg^2}=\frac{1}{2h_t^2h_qh_g^2}\Big(-\bar{p}_{i-1,j-1,r-1}+2\bar{p}_{i-1,j-1,r}-$$
$$-\bar{p}_{i-1,j-1,r+1}+\bar{p}_{i-1,j+1,r-1}-2\bar{p}_{i-1,j+1,r}+\bar{p}_{i-1,j+1,r+1}+$$
$$+2\bar{p}_{i,j-1,r-1}-4\bar{p}_{i,j-1,r}+2\bar{p}_{i,j-1,r+1}-2\bar{p}_{i,j+1,r-1}+4\bar{p}_{i,j+1,r}-$$
$$-2\bar{p}_{i,j+1,r+1}-\bar{p}_{i+1,j-1,r-1}+2\bar{p}_{i+1,j-1,r}-$$
$$-\bar{p}_{i+1,j-1,r+1}+\bar{p}_{i+1,j+1,r-1}-2\bar{p}_{i+1,j+1,r}+\bar{p}_{i+1,j+1,r+1}\Big);$$

$$S^{(6)}_{2,0}(\tilde{p})_{t^2q^2g^2}=\frac{1}{h_t^2h_q^2h_g^2}\Big(\bar{p}_{i-1,j-1,r-1}-2\bar{p}_{i-1,j-1,r}+$$
$$+\bar{p}_{i-1,j-1,r+1}-2\bar{p}_{i-1,j,r-1}+4\bar{p}_{i-1,j,r}-2\bar{p}_{i-1,j,r+1}+$$
$$+\bar{p}_{i-1,j+1,r-1}-2\bar{p}_{i-1,j+1,r}+\bar{p}_{i-1,j+1,r+1}-2\bar{p}_{i,j-1,r-1}+$$

$$+4\bar{p}_{i,j-1,r}-2\bar{p}_{i,j-1,r+1}+4\bar{p}_{i,j,r-1}-8\bar{p}_{i,j,r}+$$
$$+4\bar{p}_{i,j,r+1}-2\bar{p}_{i,j+1,r-1}+4\bar{p}_{i,j+1,r}-2\bar{p}_{i,j+1,r+1}+$$
$$+\bar{p}_{i+1,j-1,r-1}-2\bar{p}_{i+1,j-1,r}+\bar{p}_{i+1,j-1,r+1}-2\bar{p}_{i+1,j,r-1}+$$
$$+4\bar{p}_{i+1,j,r}-2\bar{p}_{i+1,j,r+1}+\bar{p}_{i+1,j+1,r-1}-2\bar{p}_{i+1,j+1,r}+\bar{p}_{i+1,j+1,r+1}\Big);$$

$$\bar{p}_{i,j,r}=\frac{1}{h_t h_q h_g}\int\limits_{-h_t/2}^{h_t/2}\int\limits_{-h_q/2}^{h_q/2}\int\limits_{-h_g/2}^{h_g/2}\Big(p+p'_t\tau+p'_q\nu+p'_q\zeta+$$
$$+\frac{1}{2}p''_{t^2}\tau^2+\frac{1}{2}p''_{q^2}\nu^2+\frac{1}{2}p''_{g^2}\zeta^2+p''_{tq}\tau\nu+p''_{tg}\tau\zeta+$$
$$+p''_{qg}\nu\zeta+\frac{1}{2}p'''_{t^2q}\tau^2\nu+\frac{1}{2}p'''_{tq^2}\tau\nu^2+\frac{1}{2}p'''_{t^2g}\tau^2\zeta+$$
$$+\frac{1}{2}p'''_{tg^2}\tau\zeta^2+\frac{1}{2}p'''_{q^2g}\nu^2\zeta+\frac{1}{2}p'''_{qg^2}\nu\zeta^2+p'''_{tqg}\tau\nu\zeta+$$
$$+\frac{1}{4}p^{(4)}_{t^2q^2}\tau^2\nu^2+\frac{1}{4}p^{(4)}_{t^2g^2}\tau^2\zeta^2+\frac{1}{4}p^{(4)}_{q^2g^2}\nu^2\zeta^2+\frac{1}{2}p^{(4)}_{t^2qg}\tau^2\nu\zeta+$$
$$+\frac{1}{2}p^{(4)}_{tq^2g}\tau\nu^2\zeta+\frac{1}{2}p^{(4)}_{tqg^2}\tau\nu\zeta^2+\frac{1}{4}p^{(5)}_{t^2q^2g}\tau^2\nu^2\zeta+\frac{1}{4}p^{(5)}_{t^2qg^2}\tau^2\nu\zeta^2+$$
$$+\frac{1}{4}p^{(5)}_{tq^2g^2}\tau\nu^2\zeta^2+\frac{1}{8}p^{(6)}_{t^2q^2g^2}\tau^2\nu^2\zeta^2\Big)d\tau d\nu d\zeta=p+\frac{1}{24}p''_{t^2}h_t^2+$$
$$+\frac{1}{24}p''_{q^2}h_q^2+\frac{1}{24}p''_{g^2}h_g^2+\frac{1}{576}p^{(4)}_{t^2q^2}h_t^2h_q^2+\frac{1}{576}p^{(4)}_{t^2g^2}h_t^2h_g^2+$$
$$+\frac{1}{576}p^{(4)}_{q^2g^2}h_q^2h_g^2+\frac{1}{13824}p^{(6)}_{t^2q^2g^2}h_t^2h_q^2h_g^2+O\left(h^6\right);$$

$$\bar{p}_{i+1,j+1,r-1}=p+p'_th_t+p'_qh_q-p'_gh_g+\frac{13}{24}p''_{t^2}h_t^2+$$
$$+\frac{13}{24}p''_{q^2}h_q^2+\frac{13}{24}p''_{g^2}h_g^2+p''_{tq}h_th_q-p''_{tg}h_th_g-p''_{qg}h_qh_g+$$
$$+\frac{13}{24}p'''_{t^2q}h_t^2h_q+\frac{13}{24}p'''_{tq^2}h_th_q^2-\frac{13}{24}p'''_{t^2g}h_t^2h_g+\frac{13}{24}p'''_{tg^2}h_th_g^2-$$
$$-\frac{13}{24}p'''_{q^2g}h_q^2h_g+\frac{13}{24}p'''_{qg^2}h_qh_g^2-p'''_{tqg}h_th_qh_g+$$
$$+\frac{169}{576}p^{(4)}_{t^2q^2}h_t^2h_q^2+\frac{169}{576}p^{(4)}_{t^2g^2}h_t^2h_g^2+\frac{169}{576}p^{(4)}_{q^2g^2}h_q^2h_g^2-$$
$$-\frac{13}{24}p^{(4)}_{t^2qg}h_t^2h_qh_g-\frac{13}{24}p^{(4)}_{tq^2g}h_th_q^2h_g+\frac{13}{24}p^{(4)}_{tqg^2}h_th_qh_g^2-$$
$$-\frac{169}{576}p^{(5)}_{t^2q^2g}h_t^2h_q^2h_g+\frac{169}{576}p^{(5)}_{t^2qg^2}h_t^2h_qh_g^2+$$
$$+\frac{169}{576}p^{(5)}_{tq^2g^2}h_th_q^2h_g^2+\frac{2197}{13824}p^{(6)}_{t^2q^2g^2}h_t^2h_q^2h_g^2+O\left(h^6\right);$$

$$\bar{p}_{i,j+1,r}=p+p'_qh_q+\frac{1}{24}p''_{t^2}h_t^2+\frac{13}{24}p''_{q^2}h_q^2+\frac{1}{24}p''_{g^2}h_g^2+$$
$$+\frac{1}{24}p'''_{t^2q}h_t^2h_q+\frac{1}{24}p'''_{qg^2}h_qh_g^2+\frac{13}{576}p^{(4)}_{t^2q^2}h_t^2h_q^2+$$
$$+\frac{1}{576}p^{(4)}_{t^2g^2}h_t^2h_g^2+\frac{13}{576}p^{(4)}_{q^2g^2}h_q^2h_g^2+\frac{1}{576}p^{(5)}_{t^2qg^2}h_t^2h_qh_g^2+$$
$$+\frac{13}{13824}p^{(6)}_{t^2q^2g^2}h_t^2h_q^2h_g^2+O\left(h^6\right);$$

and so on...

After taking into consideration the above mentioned expression, it is not difficult to prove that

$$p(t,q,g)-S_{2,0}(\bar{p},t,q,g)=-\frac{h_t^2}{6}\Big(p''_{t^2}+p'''_{t^2q}\nu+p'''_{t^2q}\zeta+$$
$$+\frac{1}{2}p^{(4)}_{t^2q^2}\nu^2+\frac{1}{2}p^{(4)}_{t^2g^2}\zeta^2+p^{(4)}_{t^2qg}\nu\zeta+\frac{1}{2}p^{(5)}_{t^2q^2g}\nu^2\zeta+\frac{1}{2}p^{(5)}_{t^2qg^2}\nu\zeta^2+$$
$$+\frac{1}{4}p^{(6)}_{t^2q^2g^2}\nu^2\zeta^2\Big)-\frac{h_q^2}{6}\Big(p''_{q^2}+p'''_{tq^2}\tau+p'''_{q^2g}\zeta+\frac{1}{2}p^{(4)}_{t^2q^2}\tau^2+$$
$$+\frac{1}{2}p^{(4)}_{q^2g^2}\zeta^2+p^{(4)}_{tq^2g}\tau\zeta+\frac{1}{2}p^{(5)}_{t^2q^2g}\tau^2\zeta+\frac{1}{2}p^{(5)}_{tq^2g^2}\tau\zeta^2+$$
$$+\frac{1}{4}p^{(6)}_{t^2q^2g^2}\tau^2\zeta^2\Big)-\frac{h_g^2}{6}\Big(p''_{g^2}+p'''_{tg^2}\tau+p'''_{qg^2}\nu+\frac{1}{2}p^{(4)}_{t^2g^2}\tau^2+$$
$$+\frac{1}{2}p^{(4)}_{q^2g^2}\nu^2+p^{(4)}_{tqg^2}\tau\nu+\frac{1}{2}p^{(5)}_{t^2q^2g}\nu^2\zeta+\frac{1}{2}p^{(5)}_{t^2qg^2}\nu\zeta^2+$$
$$+\frac{1}{4}p^{(6)}_{t^2q^2g^2}\nu^2\zeta^2\Big)-\frac{h_t^2h_q^2}{36}\Big(p^{(4)}_{t^2q^2}+p^{(5)}_{t^2q^2g}\zeta+\frac{1}{2}p^{(6)}_{t^2q^2g^2}\zeta^2\Big)-$$
$$-\frac{h_t^2h_g^2}{36}\Big(p^{(4)}_{t^2g^2}+p^{(5)}_{t^2qg^2}\nu+\frac{1}{2}p^{(6)}_{t^2q^2g^2}\nu^2\Big)-\frac{h_q^2h_g^2}{36}\Big(p^{(4)}_{q^2g^2}+$$
$$+p^{(5)}_{tq^2g^2}\tau+\frac{1}{2}p^{(6)}_{t^2q^2g^2}\tau^2\Big)-\frac{h_t^2h_q^2h_g^2}{216}p^{(6)}_{t^2q^2g^2}+O\left(h^6\right).$$

On the other hand we have:

$$p''_{t^2}(t,q,g)=p''_{t^2}+p'''_{t^2q}\nu+p'''_{t^2g}\zeta+\frac{1}{2}p^{(4)}_{t^2q^2}\nu^2+$$
$$+\frac{1}{2}p^{(4)}_{t^2g^2}\zeta^2+p^{(4)}_{t^2qg}\nu\zeta+\frac{1}{2}p^{(5)}_{t^2q^2g}\nu^2\zeta+$$
$$+\frac{1}{2}p^{(5)}_{t^2qg^2}\nu\zeta^2+\frac{1}{4}p^{(6)}_{t^2q^2g^2}\nu^2\zeta^2+O\left(h^4\right);$$

$$p''_{q^2}(t,q,g)=p''_{q^2}+p'''_{tq^2}\tau+p'''_{q^2g}\zeta+\frac{1}{2}p^{(4)}_{t^2q^2}\tau^2+$$
$$+\frac{1}{2}p^{(4)}_{q^2g^2}\zeta^2+p^{(4)}_{tq^2g}\tau\zeta+\frac{1}{2}p^{(5)}_{t^2q^2g}\tau^2\zeta+\frac{1}{2}p^{(5)}_{tq^2g^2}\tau\zeta^2+$$
$$+\frac{1}{4}p^{(6)}_{t^2q^2g^2}\tau^2\zeta^2+O\left(h^4\right);$$

$$p''_{g^2}(t,q,g)=p''_{g^2}+p'''_{tg^2}\tau+p'''_{qg^2}\nu+\frac{1}{2}p^{(4)}_{t^2g^2}\tau^2+$$
$$+\frac{1}{2}p^{(4)}_{q^2g^2}\nu^2+p^{(4)}_{tqg^2}\tau\nu+\frac{1}{2}p^{(5)}_{t^2qg^2}\tau^2\nu+\frac{1}{2}p^{(5)}_{tq^2g^2}\tau\nu^2+$$
$$+\frac{1}{4}p^{(6)}_{t^2q^2g^2}\tau^2\nu^2+O\left(h^4\right);$$

$$p^{(4)}_{t^2q^2}(t,q,g)=p^{(4)}_{t^2q^2}+p^{(5)}_{t^2q^2g}\zeta+\frac{1}{2}p^{(6)}_{t^2q^2g^2}\zeta^2+O\left(h^2\right);$$

$$p^{(4)}_{t^2g^2}(t,q,g)=p^{(4)}_{t^2g^2}+p^{(5)}_{t^2qg^2}\nu+\frac{1}{2}p^{(6)}_{t^2q^2g^2}\nu^2+O\left(h^2\right);$$

$$p^{(4)}_{q^2g^2}(t,q,g)=p^{(4)}_{q^2g^2}+p^{(5)}_{tq^2g^2}\tau+\frac{1}{2}p^{(6)}_{t^2q^2g^2}\tau^2+O\left(h^2\right);$$

$$p^{(6)}_{t^2q^2g^2}(t,q,g)=p^{(6)}_{t^2q^2g^2}+O(h).$$

It means that we can prove the correctness of the expression (3). The theorem is proved.

**Corollary 1.** Under $h_t \to 0$, $h_q \to 0$, $h_g \to 0$ for $\forall p(t,q,g) \in C^{2,2,2}$ will be correct

$$\left\|p(t,q,g) - S_{2,0}(\bar{p},t,q,g)\right\| = \frac{h_t^2}{6}\left\|p''_{t^2}(t,q,g)\right\| + \frac{h_q^2}{6}\left\|p''_{q^2}(t,q,g)\right\| + \frac{h_g^2}{6}\left\|p''_{g^2}(t,q,g)\right\| + \frac{h_t^2 h_q^2}{36}\left\|p^{(4)}_{t^2q^2}(t,q,g)\right\| + \frac{h_t^2 h_g^2}{36}\left\|p^{(4)}_{t^2g^2}(t,q,g)\right\| + \frac{h_q^2 h_g^2}{36}\left\|p^{(4)}_{q^2g^2}(t,q,g)\right\| + \frac{h_t^2 h_q^2 h_g^2}{216}\left\|p^{(6)}_{t^2q^2g^2}(t,q,g)\right\| + O(h^6).$$

Let us carry out the standard spline evaluation (1).

**Theorem 2.** For the spline $S_{2,0}(p,t,q,g)$ the evaluation $\left\|S_{2,0}(p,t,q,g)\right\| = \left\|p(t,q,g)\right\|$ is fulfilled.

**Proof.** Let us consider the representation $S_{2,0}(p,t,q,g)$ in the form of

$$S_{2,0}(p,t,q,g) = \frac{1}{512}\Big((1-x)^2(1-y)^2(1-z)^2 p_{i-1,j-1,r-1} + (1-x)^2(1-y)^2(6-2z)p_{i-1,j-1,r} + (1-x)^2(1-y)^2(1+z)^2 p_{i-1,j-1,r+1} + (1-x)^2(6-2y)(1-z)^2 p_{i-1,j,r-1} + (1-x)^2(6-2y)(6-2z)p_{i-1,j,r} + (1-x)^2(6-2y)(1+z)^2 p_{i-1,j,r+1} + (1-x)^2(1+y)^2(1-z)^2 p_{i-1,j+1,r-1} + (1-x)^2(1+y)^2(6-2z)p_{i-1,j+1,r} + (1-x)^2(1+y)^2(1+z)^2 p_{i-1,j+1,r+1} + (6-2x)(1-y)^2(1-z)^2 p_{i,j-1,r-1} + (6-2x)(1-y)^2(6-2z)p_{i,j-1,r} + (6-2x)(1-y)^2(1+z)^2 p_{i,j-1,r+1} + (6-2x)(6-2y)(1-z)^2 p_{i,j,r-1} + (6-2x)(6-2y)(6-2z)p_{i,j,r} + (6-2x)(6-2y)(1+z)^2 p_{i,j,r+1} + (6-2x)(1+y)^2(1-z)^2 p_{i,j+1,r-1} + (6-2x)(1+y)^2(6-2z)p_{i,j+1,r} + (6-2x)(1+y)^2(1+z)^2 p_{i,j+1,r+1} + (1+x)^2(1-y)^2(1-z)^2 p_{i+1,j-1,r-1} + (1+x)^2(1-y)^2(6-2z)p_{i+1,j-1,r} + (1+x)^2(1-y)^2(1+z)^2 p_{i+1,j-1,r+1} + (1+x)^2(6-2y)(1-z)^2 p_{i+1,j,r-1} + (1+x)^2(6-2y)(6-2z)p_{i+1,j,r} + (1+x)^2(6-2y)(1+z)^2 p_{i+1,j,r+1} + (1+x)^2(1+y)^2(1-z)^2 p_{i+1,j+1,r-1} + (1+x)^2(1+y)^2(6-2z)p_{i+1,j+1,r} + (1+x)^2(1+y)^2(1+z)^2 p_{i+1,j+1,r+1}\Big),$$

where $x = \frac{2}{h_t}(t - ih_t)$, $|x| \le 1$;

$y = \frac{2}{h_q}(q - jh_q)$, $|y| \le 1$; $z = \frac{2}{h_g}(q - rh_g)$, $|z| \le 1$.

Then

$$\left\|S_{2,0}(p,t,q,g)\right\| \le \frac{1}{512}\left\|p(t,q,g)\right\| \max_{|x|\le1;|y|\le1;|z|\le1} A(x,y,z),$$

where

$$A(x,y,z) = \left|(1-x)^2(1-y)^2(1-z)^2\right| + \left|(1-x)^2(1-y)^2(6-2z)\right| + \left|(1-x)^2(1-y)^2(1+z)^2\right| + \left|(1-x)^2(6-2y)(1-z)^2\right| + \left|(1-x)^2(6-2y)(6-2z)\right| + \left|(1-x)^2(6-2y)(1+z)^2\right| + \left|(1-x)^2(1+y)^2(1-z)^2\right| + \left|(1-x)^2(1+y)^2(6-2z)\right| + \left|(1-x)^2(1+y)^2(1+z)^2\right| + \left|(6-2x)(1-y)^2(1-z)^2\right| + \left|(6-2x)(1-y)^2(6-2z)\right| + \left|(6-2x)(1-y)^2(1+z)^2\right| + \left|(6-2x)(6-2y)(1-z)^2\right| + \left|(6-2x)(6-2y)(6-2z)\right| + \left|(6-2x)(6-2y)(1+z)^2\right| + \left|(6-2x)(1+y)^2(1-z)^2\right| + \left|(6-2x)(1+y)^2(6-2z)\right| + \left|(6-2x)(1+y)^2(1+z)^2\right| + \left|(1+x)^2(1-y)^2(1-z)^2\right| + \left|(1+x)^2(1-y)^2(6-2z)\right| + \left|(1+x)^2(1-y)^2(1+z)^2\right| + \left|(1+x)^2(6-2y)(1-z)^2\right| + \left|(1+x)^2(6-2y)(6-2z)\right| + \left|(1+x)^2(6-2y)(1+z)^2\right| + \left|(1+x)^2(1+y)^2(1-z)^2\right| + \left|(1+x)^2(1+y)^2(6-2z)\right| + \left|(1+x)^2(1+y)^2(1+z)^2\right|.$$

After taking into consideration the fact that the function $A(x,y,z)$ is even, it will be enough to examine it for $x,y,z \in [0;1]$ in order to define its it maximum. Knowing the fact that the expression under the module sign is greater than 0, it is not difficult to prove that $A(x,y,z) = 512$. Thus

$$\left\|S_{2,0}(p,t,q,g)\right\| \le \left\|p(t,q,g)\right\|. \quad (4)$$

On the other hand, for any special case $\left\|S_{2,0}(p,t,q,g)\right\| \ge \left\|S_{2,0}(\tilde{p},t,q,g)\right\|$.

Then for $x=0$, $y=0$, $z=0$, we have

$$S_{2,0}(\tilde{p},ih_t,jh_q,rh_g)=\frac{1}{512}\left(\tilde{p}_{i-1,j-1,r-1}+6\tilde{p}_{i-1,j-1,r}+\right.$$
$$+\tilde{p}_{i-1,j-1,r+1}+6\tilde{p}_{i-1,j,r-1}+36\tilde{p}_{i-1,j,r}+6\tilde{p}_{i-1,j,r+1}+$$
$$+\tilde{p}_{i-1,j+1,r-1}+6\tilde{p}_{i-1,j+1,r}+\tilde{p}_{i-1,j+1,r+1}+6\tilde{p}_{i,j-1,r-1}+$$
$$+36\tilde{p}_{i,j-1,r}+6\tilde{p}_{i,j-1,r+1}+36\tilde{p}_{i,j,r-1}+$$
$$+216\tilde{p}_{i,j,r}+36\tilde{p}_{i,j,r+1}+6\tilde{p}_{i,j+1,r-1}+36\tilde{p}_{i,j+1,r}+$$
$$+6\tilde{p}_{i,j+1,r+1}+\tilde{p}_{i+1,j-1,r-1}+6\tilde{p}_{i+1,j-1,r}+$$
$$+\tilde{p}_{i+1,j-1,r+1}+6\tilde{p}_{i+1,j,r-1}+36\tilde{p}_{i+1,j,r}+$$
$$\left.+6\tilde{p}_{i+1,j,r+1}+\tilde{p}_{i+1,j+1,r-1}+6\tilde{p}_{i+1,j+1,r}+\tilde{p}_{i+1,j+1,r+1}\right).$$

If $\tilde{p}_{a,b,c}=\left\|p(t,q,g)\right\|$,

$a=\overline{i-1;i+1}$; $b=\overline{j-1;j+1}$; $c=\overline{r-1;r+1}$,

then $\left\|S_{2,0}(p,t,q,g)\right\| \ge \left\|S_{2,0}(p^*,t,q,g)\right\| \ge$

$\ge \left|S_{2,0}(\tilde{p},0,0,0)\right| = \left\|p(t,q,g)\right\|$.

It means that with regard to (4), we can find out that

$\left\|S_{2,0}(p,t,q,g)\right\| = \left\|p(t,q,g)\right\|$.

The theorem is proved.

**Corollary 2.** For $\forall p(t,q,g)\in C^{2,2,2}$ there is

$$\left\|p(t,q,g)-S_{2,0}(p,t,q,g)\right\| \le \frac{h_t^2}{6}\left\|p''_{t^2}(t,q,g)\right\|+$$
$$+\frac{h_q^2}{6}\left\|p''_{q^2}(t,q,g)\right\|+\frac{h_g^2}{6}\left\|p''_{g^2}(t,q,g)\right\|+$$
$$+\frac{h_t^2h_q^2}{36}\left\|p^{(4)}_{t^2q^2}(t,q,g)\right\|+\frac{h_t^2h_g^2}{36}\left\|p^{(4)}_{t^2g^2}(t,q,g)\right\|+$$
$$+\frac{h_q^2h_g^2}{36}\left\|p^{(4)}_{q^2g^2}(t,q,g)\right\|+\frac{h_t^2h_q^2h_g^2}{216}\left\|p^{(6)}_{t^2q^2g^2}(t,q,g)\right\|+$$
$$+\varepsilon\left\|p(t,q,g)\right\|+O(h^6).$$

The statement follows from the theorem 1 and 2 as well as the corollary 2.

**Corollary 3.** Let us $\{p_{i,j,r};i,j,r\in Z\}$ – is the meaning of the empirical function of distribution density of three –dimensional random variable probabilities, where

$$h_th_qh_g\sum_{i\in Z}\sum_{j\in Z}\sum_{r\in Z}p_{i,j,r}=1,$$

$p(t,q,g)$ – is the correspondent theorem theoretical function of the distribution density,

$$\overline{p}_{i,j,r}=\frac{1}{h_th_qh_g}\int_{(i-1/2)h_t}^{(i+1/2)h_t}\int_{(j-1/2)h_q}^{(j+1/2)h_q}\int_{(r-1/2)h_g}^{(r+1/2)h_g}p(t,q,g)\,dtdqdg,$$

$$\varepsilon_{i,j,r}=\overline{p}_{i,j,r}-p_{i,j,r},\quad \varepsilon=\max_{i,j,r}\{\varepsilon_{i,j,r}\},$$

then $\left\|S_{2,0}(p,t,q,g)\right\|=1$ and for $\forall\, p(t,q,g)\in C^{2,2,2}$

$$\left\|p(t,q,g)-S_{2,0}(p,t,q,g)\right\| \le \frac{h_t^2}{6}\left\|p''_{t^2}(t,q,g)\right\|+$$
$$+\frac{h_q^2}{6}\left\|p''_{q^2}(t,q,g)\right\|+\frac{h_g^2}{6}\left\|p''_{g^2}(t,q,g)\right\|+\frac{h_t^2h_q^2}{36}\left\|p^{(4)}_{t^2q^2}(t,q,g)\right\|+$$
$$+\frac{h_t^2h_g^2}{36}\left\|p^{(4)}_{t^2g^2}(t,q,g)\right\|+\frac{h_q^2h_g^2}{36}\left\|p^{(4)}_{q^2g^2}(t,q,g)\right\|+$$
$$+\frac{h_t^2h_q^2h_g^2}{216}\left\|p^{(6)}_{t^2q^2g^2}(t,q,g)\right\|+\varepsilon+O(h^6).$$

The results obtained testify to the high-level approximating properties of spline $S_{2,0}(p,t,q,g)$ and give the right to recommend to use them in the field of data observation processing, visualization of space models of different kinds as well as in the area of the probabilistic evaluation of the density distribution function of three-dimensional random variable realization probabilities.

### References

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П.О. Приставка

Відновлення функції щільності розподілу трьох змінних поліноміальним сплайном на основі В-сплайнів

Досліджено асимптотичні властивості поліноміального сплайну трьох змінних на основі В-сплайнів другого порядку. Сформульовано і доведено відповідні теореми, наведено наслідки.

Ф.А. Приставка

Восстановление функции плотности распределения трех переменных полиномиальным сплайном на основе В-сплайнов

Исследованы асимптотические свойства полиномиального сплайна трех переменных на основе В-сплайнов второго порядка. Сформулированы и доказаны соответствующие теоремы и следствия.