## PHYSICO-MATHEMATICAL SCIENCES

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# CALCULATION OF GAMMA FUNCTIONS SIGNIFICANCES

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*The existing variants of the Euler's gamma function presentation are investigated in view of their application for calculations of this function values. The accuracy of calculations has been investigated experimentally depending on these variants parameters.*

## Introduction

The solutions of different problems of physics and engineering can frequently be based on Euler's gamma functions in their direct or indirect form. In particular, the Laplace and Frenel integrals, Hermite and Laguerre spherical and cylindrical functions etc. are presented by these functions.

One of the reasons of gamma functions wide implementation in practice is that they can be used for obtaining a solution of one of the simplest difference equation. The most importance of these functions is linked with solving homogeneous and non-homogeneous linear difference equations with coefficients in the form of rational functions or functions containing factorial series.

There are many tables with tabulated values of Euler's gamma functions (for example, see [1]). Unfortunately, practical use of these tables is frequently connected with some inconveniences. That is why the computer oriented evaluation of these special functions is an actual problem.

## Statement of a problem

The objective of this article is to investigate different methods of Euler's gamma function evaluation and developing the corresponding algorithm.

## Basic material

As it is known the Euler's gamma function is determined as follows

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt\,.$$

Theoretical-functional analysis of gamma function reflects one of its most important properties, namely: it doesn't satisfy any differential equation. Therefore, the gamma function has higher order of transcendency with respect to functions, that satisfy differential equations, and the gamma function calculation is possible only with the use of its approximate representations. The following problems are actual in the process of gamma functions evaluation:

– which formula must be selected regarding the rate *f* series convergent;

– determination of the parameter $\theta$ value, that is contained in remainder of series for gamma function computations;

– what is the interval of computations of gamma function significances;

– how many terms of series must be kept for required accuracy providing.

Let us consider the formulated problems in more detailed form.

## Some formulas for computation the gamma function

There are different formulas for approximate representation of gamma function. We shall start their consideration with the asymptotic Stirling decomposition that has the following structure

$$\Gamma(x)=\sqrt{2\pi}\,x^{x-\frac{1}{2}}e^{-x}\left(1+\frac{1}{12x}+\frac{1}{288x^2}-\right.$$
$$-\frac{139}{51840x^3}-\frac{571}{2488320x^4}+\frac{163879}{209018880x^5}+$$
$$\left.+\frac{5246819}{75246796800x^6}-\frac{534703531\cdot\theta}{902961561600x^7}\right), \quad (1)$$

$0<\theta<1$.

In reality this decomposition was presented by Moivre, but Stirling reformulated it in its final form by determining the constant factor. Coefficients of formula (1) were determined by Davis; who checked them and proved their correctness [2].

Frequently Stirling's series is associated with the following form [3]:

$$\ln\Gamma(z)=\left(z-\frac{1}{2}\right)\ln z+\frac{1}{2}\ln 2\pi-z+$$
$$+\left(1+\sum_{n=1}^{m-1}\frac{B_{2n}}{2n(2n-1)z^{2n-1}}+\right.$$
$$\left.+\frac{B_{2m}}{2m(2m-1)(z+\theta)^{2n-1}}\right), \quad (2)$$

$0<\theta<1$,

where $B_{2n}$ are Bernoulli numbers.

The Bernoulli numbers can often be the coefficients of power series decomposition of different elementary functions. For example, the following equality is true

$$\frac{x}{e^x-1}=\sum_{k=0}^{\infty}B_k\frac{x^k}{k!}.$$

This decomposition is called the generating function for the Bernoulli numbers. The next dependence is useful too:

$$B_0=1;$$

$$\sum_{k=0}^{n-1}C_n^kB_k=0.$$

There are detailed tables for the Bernoulli numbers. Let us determine several first Bernoulli numbers:

$$B_0=1;$$

$$B_1=-\frac{1}{2};$$

$$B_2=\frac{1}{6};$$

$$B_3=0;$$

$$B_4=-\frac{1}{30};$$

$$B_5=0;$$

$$B_6=\frac{1}{42};\quad \ldots$$

By exponentiation of (2) we will get

$$\Gamma(z)=\sqrt{2\pi}\,z^{z-\frac{1}{2}}\exp\left(-z+\sum_{n=1}^{m-1}\frac{B_{2n}}{2n(2n-1)z^{2n-1}}+\right.$$

$$\left.+\frac{B_{2m}}{2m(2m-1)(z+\theta)^{2m-1}}\right). \quad (3)$$

For the real argument $x$ formula (3) will have a form [4]

$$\Gamma(x+1)=\sqrt{2\pi}x^{x+\frac{1}{2}}\exp\left(-x+\sum_{n=1}^{m-1}\frac{B_{2n}}{2n(2n-1)x^{2n-1}}+\right.$$

$$\left.+\frac{B_{2m}}{2m(2m-1)(x+\theta)^{2m-1}}\right). \quad (4)$$

The formula

$$\Gamma(x+1)=\sqrt{2\pi}\,x^{x+\frac{1}{2}}e^{-x+\frac{1}{12(x+\theta)}}, \quad (5)$$

$$0<\theta<1,$$

that contains only first term of series (4), is called the Sonin's formula. In general, having in mind a great contribution of Sonin to research of series (4), in future this series will be called by us as Stirling-Sonin series.

Let us introduce the gamma function, that is determined as

$$\psi(x)=\sum_{r=1}^{\infty}\frac{x}{r(r+x)}-C, \qquad \psi(0)=-C,$$

where $C=0{,}5772156649\ldots$ is the Euler's constant.

As it is known [5], the digamma function is connected with gamma function by relation:

$$\frac{d}{dx}\ln\left[\Gamma(x+1)\right]=\psi(x).$$

It can be shown [6], that the digamma function application allows to determine the following representation of gamma function

$$\frac{1}{\Gamma(z+1)}=e^{Cz}\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)e^{-\frac{z}{k}}, \quad (6)$$

where $C=0{,}5772156649\ldots$ is the Euler's constant.

Sometimes the significances of gamma function can be calculated on the base of its decomposition by Chebyshev polynomials [7] that is:

$$\Gamma(x+1)=\sum_{n=0}^{\infty}a_nT_n^*(x),$$

$$0\le x\le 1. \quad (7)$$

Some other decompositions of the function $\Gamma(z)$ and its logarithm $\ln\Gamma(z)$, which were worked out by Forsait, Pierson, Kummer, Legendre and Burge, can be found in [2]. Legendre decomposition and the others that are not included into [2], are presented in [8].

### Stirling series application for evaluation of significances of gamma functions

For evaluating the significances of gamma functions on the basis of Stirling series the formula (1) has been applied. The corresponding to (1) program has been elaborated by Pascal programming language. Gamma function values, obtained by this program, were compared with the corresponding values, found on the basis of mathematical package MathCAD. In the process of computations the following problems were defined:

– determination of the significance of parameter $\theta$, that is contained in remainder of series (1);

– choosing the interval of computations of gamma function significances;

– estimating the number of terms of series (1) for the gamma function calculation with required level of accuracy.

On the first stage only two terms of (1) were kept and the value of argument was limited by interval [1; 2].

The obtained results are presented in tab. 1.

*Table 1*

| Significance of parameter $\theta$ | Maximum relative error $\delta$, % |
|---|---|
| 0,0001 | 7,78553 |
| 0,5 | 3,94406 |
| 0,9999 | 0,10259 |

Analysis of tab. 1 shows that the highest meaning of error was obtained for the following significance of parameter $\theta = 0{,}9999$. Further we fix the significance of this parameter taking $\theta = 0{,}9999$.

At the next stage of this research the interval of calculating the significances of gamma function was determined. Necessity of this choice is explained by the following factors.

First of all, let us notice the fact of getting the highest meaning of relative errors at a left boundary point of interval. It can be explained by the asymptotic behavior of the Stirling formula, so its accuracy increases if its argument increases. Taking into account gamma function properties we are able to recalculate its significances for any interval. At the same time the process of gamma function evaluation for large values of argument requires calculating the factorial products of big natural numbers that demands computations with many significant digits. Under these circumstances the problem of choosing the optimal interval is natural.

Due to the fact that statement of this problem and its investigation is not the aim of this article let us limit its research by evaluating the gamma function in frames of the following intervals [1, 2], [5, 6], [10, 11]. At that time for comparison the calculated significances were recalculated for the interval [1, 2]. Obtained results are summarized in tab. 2.

*Table 2*

| Variation interval of argument $[a, b]$ | Maximum relative error $\delta$, % |
|---|---|
| [1, 2] | 0,10259 |
| [5, 6] | 0,0117 |
| [10, 11] | 0,00326 |

Analysis of the obtained results shows, that the smallest relative error corresponds to interval [10, 11], so in future we shall evaluate the gamma function significances just inside this interval.

During the next stage of calculations the rate of Stirling series convergence has been investigated. For this purpose we realized calculations with 2, 4 and 8 first terms of series. The obtained results arte collected in tab. 3.

*Table 3*

| Number of series terms | Maximum relative error $\delta$, % |
|---|---|
| 2 | 0,00326 |
| 4 | $1{,}524 \times 10^{-6}$ |
| 8 | $5{,}841 \times 10^{-9}$ |

Analysing results of carried out computations it is possible to conclude, that the gamma function significances evaluating by formula (1) inside the interval $[10, 11]$ with the use of $\theta = 0{,}9999$ provides the relative accuracy 0,003 % by taking 2 first terms of series, the relative accuracy $1{,}5 \times 10^{-6}$ % for 4 kept first series terms, and the relative accuracy $5{,}8 \times 10^{-9}$ % for taken 8 first terms of a series. This high level of accuracy is required because of the necessity to apply the gamma function significances in further computations.

**Stirling-Sonin series in the problem of evaluating the gamma functions significances**

For evaluating the gamma functions significances using the Stirling-Sonin series formula (4) has been applied. Like at previous stage of research, for the gamma functions significances evaluating the Pascal computer program has been elaborated and significances of gamma function, obtained by this program, were compared with the corresponding values, found on the base of mathematical package MathCAD. In the process of computations like at previous stage of research the following problems were defined:

– determination of the significance of parameter $\theta$, that is contained in remainder of series (4);

– choosing the interval of computations of gamma function significances;

– estimating the number of terms of series (4) for the gamma function calculation with required level of accuracy.

The first step of calculation has been realized on the basis of Sonin formula (5) inside the interval $[1, 2]$.

Results of calculations for values of parameter $\theta = 0{,}0001$, $\theta = 0{,}5$, $\theta = 0{,}9999$ are presented in tab. 4.

*Table 4*

| Significance of parameter $\theta$ | Maximum relative error $\delta$, % |
|---|---|
| 0,0001 | 0,22661 |
| 0,5 | 2,51834 |
| 0,9999 | 3,86269 |

The results presented in this table demonstrate the best accuracy for the following value of parameter $\theta = 0{,}0001$. We fix this significance for further researches by taking it as $\theta = 0{,}0001$.

During the next stage of this research the interval of calculation of gamma function significances has been chosen. Like at previous steps, evaluation the gamma function significances was made inside the intervals $[1, 2]$, $[5, 6]$, $[10, 11]$, here the calculated significances were recalculated for the interval $[1, 2]$. Obtained results are illustrated in tab. 5.

*Table 5*

| Variation interval of argument $[a, b]$ | Maximum relative error $\delta$, % |
|---|---|
| $[1, 2]$ | 0,22661 |
| $[5, 6]$ | 0,00216 |
| $[10, 11]$ | 0,00027 |

Analysis of the obtained results shows, that the smallest relative error corresponds to interval $[10, 11]$; so in future we shall evaluate the gamma function significances just inside this interval.

At the next stage of calculations the rate of Stirling-Sonin series convergence has been investigated. For this purpose we realized calculations with 2, 4 and 8 first terms of series. The obtained results arte collected in tab. 6.

*Table 6*

| Number of series terms | Maximum relative error $\delta$, % |
|---|---|
| 2 | $7{,}794 \times 10^{-7}$ |
| 4 | $8{,}208 \times 10^{-11}$ |
| 8 | $6{,}804 \times 10^{-13}$ |

Analysing the results of performed computations it is possible to conclude, that the gamma function significances evaluating by formula (4) inside the interval $[10, 11]$ with the use of $\theta = 0{,}0001$ provides the relative accuracy $7{,}8 \times 10^{-7}$ % by taking 2 first terms of series, the relative accuracy $8{,}2 \times 10^{-11}$ % for 4 kept first series terms, and the relative accuracy $6{,}8 \times 10^{-13}$ % for 8 first terms of a series.

**Evaluating the gamma functions significances with the use of infinite product**

Evaluation of the gamma functions significances based on infinite product was performed by using formula (6). Corresponding to (6) computations the gamma functions significances were realized by using of the developed Pascal computer program. The gamma function significances, obtained by this program, were compared with the corresponding values, found on the basis of mathematical package MathCAD. Estimation of the rate of convergence was the aim of these computations for the values of argument lying in frames of the interval $[1, 2]$. The tab. 7 illustrates the determined data.

Let us notice the fact of getting the maximum relative errors at a right boundary point of interval. The made computations demonstrated very slow rate of convergence of infinite product, hence formula (6) for evaluating the gamma functions significances is not applicable for practical use.

**Evaluation of gamma functions significances by shifted Chebyshev polynomials**

The expansion of (7) in shifted Chebyshev polynomials is sometimes used for evaluating the gamma function significances. First 16 coefficients of this decomposition are presented in tab. 8.

Corresponding to (7) computations the gamma functions significances were performed by using the developed Pascal computer program. The gamma function significances, obtained by this program, like at earlier stages were compared with the corresponding values, found on the basis of mathematical package MathCAD.

*Table 7*

| Number of factors | Maximum relative error $\delta$, % |
|---|---|
| 1 000 | 5,84703 |
| 10 000 | 0,60292 |
| 100 000 | 0,06048 |

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*Table 8*

| $n$ | $a_n$ |
|---|---|
| 0 | 0,941785597795494 |
| 1 | 0,004415381324841 |
| 2 | 0,056850436815993 |
| 3 | -0,004219835396418 |
| 4 | 0,001326808181212 |
| 5 | -0,000189302452979 |
| 6 | 0,000036069253274 |
| 7 | -0,000006056761904 |
| 8 | 0,000001055629546 |
| 9 | -0,000000181196736 |
| 10 | 0,000000031177249 |
| 11 | -0,000000005354219 |
| 12 | 0,000000000919327 |
| 13 | -0,000000000157794 |
| 14 | 0,000000000027079 |
| 15 | -0,000000000004646 |

Estimation of the rate of convergence of series was the aim of these computations for the values of argument lying in frames of the interval $[1, 2]$ tab. 9 reflects the calculated results.

*Table 9*

| Number of series terms | Maximum relative error $\delta$, % |
|---|---|
| 4 | 0,146 |
| 8 | $1,157 \times 10^{-4}$ |
| 16 | $2,266 \times 10^{-8}$ |

Tab. 9 shows that the coefficients of decomposition decrease so quickly; so, in particular, the 16th coefficient of this expansion has the order $O(10^{-12})$.
At the same time, increasing the maximum coefficients of shifted Chebyshev polynomials practically balances the high order of decreasing the expansion coefficients.
Due to keeping 16 first terms of the expansion of (7) the obtained order of maximum relative error is only $10^{-8}$. Low order of expansion terms (7) decrease limits a practical application of this expansion for evaluating the gamma functions significances.

## Conclusion

The results of investigation of several variants of possible evaluating the gamma functions significances can be summarized in the form of the following conclusions.
The most preferable possibility of getting high accuracy of the gamma functions significances evaluating is the application of Stirling-Sonin series having the greatest rate of convergence.
In the process of calculations the parameter $\theta$ can be taken to be equal to $0{,}0001$. Due to asymptotic behavior of the Stirling-Sonin series and possibility to shift the argument of gamma function on any segment of numerical scale, it is advisable to make a shift of this argument on the segment $[10, 11]$.
Results of carried out experimental researches correspond well to known theoretical positions and the elaborated technique of evaluating of gamma functions significances can be recommended for practical applications.

## Literature

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В.П. Денисюк, Л.В. Рибачук
Обчислення значень гамма-функцій
Розглянуто відомі варіанти подання гамма–функції Ейлера з точки зору можливості їх використання для обчислення значень цієї функції. Експериментально досліджено вплив значень параметрів, що входять у ці подання, на точність обчислюваних значень.

В.П. Денисюк, Л.В. Рыбачук
Вычисление значений гамма-функций
Рассмотрены известные формы представления гамма–функции Эйлера с точки зрения возможности их использования для вычисления значений этой функции. Экспериментально исследовано влияние значений параметров, которые входят в эти представления, на точность вычисляемых значений.