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Viacheslav Gusynin[1]
Andrii Gusynin[2]
Helen Tachinina[3]

THE USE OF DIFFERENTIAL TRANSFORMATIONS FOR SOLVING NON-LINEAR BOUNDARY VALUE PROBLEMS

[1] Binational company «Alc?ntara Cyclone Space»
Av. L3 Norte – Ed. Finatec – Bloco H, Brasilia-DF – Brasil
[2] National Technical University of Ukraine «Igor Sikorsky KPI»
37, Peremohy ave., Kyiv, 03056, Ukraine
[3] National Aviation University
1, Kosmonavta Komarova ave., Kyiv, 03680, Ukraine
E-mails: [1]viacheslav.gusynin@gmail.com; [2]gusynin@gmail.com; [3]tachinina@mail.ru

Abstract

Purpose: *The aim of our study is comparison of method applications based on differential transformations for solving boundary value problems which are described by non-linear ordinary differential equations.* ***Methods:*** *This article reviews two approaches based on differential transformations for solving non-linear boundary value problems: the modified differential transform method and the system-analogue simulation method.* ***Results:*** *In this paper, we present results of the numerical solution of non-linear boundary value problem by methods based on differential transformations for demonstration the effectiveness and applicability of techniques. The relative error for given solutions, obtained with using first 6 discretes of differential spectra is presented.* ***Discussion:*** *Comparison of numerical solutions obtained by modified differential transform method and system-analogue simulation method with exact solution shows that both methods have good agreement with exact solution of non-linear boundary value problem for small intervals. However, application of system-analogue simulation method is preferential for big intervals, on which the boundary value problem is solved.*

Keywords: Adomian polynomials; differential transformations; modified differential transform method; non-linear boundary value problem; system-analogue simulation method.

1. Introduction

Non-linear boundary value problems occur frequently at the simulation of different problems in various fields of science and engineering, including optimal control, flight dynamics, gas dynamics, fluid mechanics, electrohydrodynamics, astrophysics, nuclear physics, quantum mechanics and others. In general, boundary value problems are described by non-linear differential equations, don't have analytical solution and are solved by various numerical and numerical-analytic methods, that proved their effectiveness [1-6]. However, application of the majority of these methods is associated with overcoming of the variety of mathematical and computational difficulties.

2. Analysis of the research and publications

One way to overcome given difficulties is the differential transform method (DTM) that was developed by academician Pukhov G.E. [7-9].

According to the DTM, the problem is solving in the image field with further translation to the original field and presentation of solution as truncated Taylor series. The main advantage of given approach is that it can be applied directly to solve of non-linear differential equations without preliminary linearization, eliminates dependence of variables from time argument, admits the possibility to obtain solution in analytic form and considerably reduces the volume of computation.

Each spectral model, obtained as result of differential transformations of the initial mathematical model, is a quasi-analogue of initial problem. Usually for the solution used one of the infinite set of quasi-analogues, which can be constructed on the base of differential transformations. Application of the system of quasi-analogues or in other words a set of spectral models, allows with equal number of discretes of differential spectrum to obtain more exact values of unknown boundary conditions and, therefore, to obtain more high solution accuracy [10]. On the base of the given approach, the system-analogue simulation method (SSM) has been proposed by Baranov [11], for solving non-linear boundary value problems based on differential transformations.

Frequently at solving of non-linear differential equations, including DTM application, occur mathematical difficulties, associated with the complex nonlinearity of equations. These difficulties can be overcoming by using of Adomian polynomials [12]. This approach uses the decomposition of non-linear differential equation into linear and non-linear parts and approximation of unknown non-linear part of equation by Adomian polynomials. Application of Adomian polynomials in the differential transform method (the modified differential transform method, MDTM) considerably simplifies solving of non-linear boundary value problems and extends the field of DTM application. The application effectiveness of modified differential transform method for solving non-linear boundary value problem can be found in [13,14].

3. Research tasks

The aim of our study is comparison of method applications based on differential transformations for solving boundary value problems which are described by non-linear ordinary differential equations.

4. Differential transformations

The differential transformations allow to replace in the mathematical model of physical process the functions $x(t)$ continuous argument t by their spectral models in the form of discrete functions $X(k)$ of integer argument $k = 0,1,2,...$.

The differential transformations of function $x(t)$ are the functional transformations of the type:

$$X(k) = \frac{H^k}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=0}, \qquad (1)$$

where $x(t)$ – the original, which represents the continuous and bounded together with all its derivatives function of real argument t; $X(k)$ – the differential image of original (differential spectrum), which represents the discrete function of integer argument $k = 0,1,2,...$ (discretes of differential spectrum); H – the scale stationary value having dimensionality of argument t and usually chosen equal to the segment $0 \le t \le H$, on which the function $x(t)$ is considered.

The inverse transformation, allows to obtain the original $x(t)$ by the image $X(k)$ in the form of Taylor series:

$$x(t) = \sum_{k=0}^{\infty}\left(\frac{t}{H}\right)^k X(k) \qquad (2)$$

or at $H = 1$:

$$x(t) = \sum_{k=0}^{\infty} X(k)t^k. \qquad (3)$$

The value of H is to be less than the radius of convergence ρ, which can be determined on the base of D'Alembert criterion:

$$\rho = \overline{\lim_{k\to\infty}}\left|\frac{X(k)}{H^k} : \frac{H(k+1)}{H^{k+1}}\right| = H\,\overline{\lim_{k\to\infty}}\left|\frac{X(k)}{X(k+1)}\right|. \qquad (4)$$

When considering the differential transformations of time function $x(t)$ in the point t_i, expressions (1) and (2) are following:

$$X(k,t_i) = \frac{H^k}{k!}\left[\frac{d^k x(t_i + \mathrm{a})}{da^k}\right]_{a=0}, \qquad (5)$$

$$x(t_i + a) = \sum_{k=0}^{\infty}\left(\frac{a}{H}\right)^k X(k,t_i), \qquad (6)$$

where a is the local argument, which value chosen in the bounds $0 \le a \le H$ or $-H \le a \le H$. Application of differential transformations (5) to the mathematical model of the object allows to construct in depend on choosing the specific value t_i, the set of corresponded spectral models (quasi-analogues of the initial problem) in the image field. In such case holds the quasi-analogue simulation, which replace the analysis of physical processes in the object by event investigations of other physical nature, which described by models that different from initial mathematical models, which are equivalent relative to necessary simulation results [10,11].

5. The modified differential transform method

Consider the operator form of a non-linear differential equation:

$$Px + Nx + Qx = c, \tag{7}$$

where $x = x(t)$; $P = \frac{d^n}{dt^n}$ is the non-linear differential operator, $n > 1$; $N = \frac{d}{dt}$ is a linear differential operator, Q represents the non-linear operator of non-linear function $f = f(x)$, c is the source term. The solution of equation (7) is defined as infinite series:

$$x(t) = \sum_{k=0}^{\infty} x_k(t). \tag{8}$$

According to the method of Adomian polynomials, the non-linear term of equation is approximated by the Adomian polynomials as infinite series:

$$Qx = \sum_{n=0}^{\infty} A_n. \tag{9}$$

Components of the Adomian polynomials are defined as:

$$A_n = \frac{1}{n!}\left\{\frac{d^n}{d\lambda^n}\left[Q\left(\sum_{l=0}^{\infty}\lambda^l x_l\right)\right]\right\}_{\lambda=0}, n \geq 0 \tag{10}$$

The Adomian polynomials for the non-linear function $f = f(x)$ are determined into the form [12]:

$$\begin{aligned}
&A_0 = f(x_0),\ A_1 = x_1 f^{(1)}(x_0),\\
&A_2 = x_2 f^{(1)}(x_0) + \frac{1}{2!}x_1^2 f^{(2)}(x_0),\\
&A_3 = x_3 f^{(1)}(x_0) + x_1 x_2 f^{(2)}(x_0) + \frac{1}{3!}x_1^3 f^{(3)}(x_0),\\
&A_4 = x_4 f^{(1)}(x_0) + \left(x_1 x_3 + \frac{1}{2!}x_2^2\right) f^{(2)}(x_0) + \frac{1}{2!}x_1^2 x_2 f^{(3)}(x_0) + \frac{1}{4!}x_1^4 f^{(4)}(x_0),\\
&A_5 = x_5 f^{(1)}(x_0) + \left(x_2 x_3 + x_1 x_4\right) f^{(2)}(x_0) + \frac{1}{2!}\left(x_1^2 x_3 + x_1 x_2^2\right) f^{(3)}(x_0) +\\
&+\frac{1}{3!}x_1^3 x_2 f^{(4)}(x_0) + \frac{1}{5!}x_1^5 f^{(5)}(x_0), \ldots
\end{aligned} \tag{11}$$

Taking into account the features of differential transformations, components of differential image of non-linear function $f(x)$ of desired differential equation are defined in the following form [13]:

$$\begin{aligned}
&F(0) = f(x(0)) = f(X(0)) = f(x_0),\\
&F(1) = \frac{d}{dt} f(x(t))\Big|_{t=0} = x'(0) f^{(1)}(x(0)) = X(1) f^{(1)}(X(0)),\\
&F(2) = X(2) f^{(1)}(X(0)) + \frac{1}{2!}(X(1))^2 f^{(2)}(X(0)),\\
&F(3) = X(3) f^{(1)}(X(0)) + X(1)X(2) f^{(2)}(X(0)) + \frac{1}{3!}(X(1))^3 f^{(3)}(X(0)),\\
&F(4) = X(4) f^{(1)}(X(0)) + (X(1)X(3) + \frac{1}{2!}(X(2))^2) f^{(2)}(X(0)) +\\
&+\frac{1}{2!}(X(1))^2 X(2) f^{(3)}(X(0)) + \frac{1}{4!}(X(1))^4 f^{(4)}(X(0)),\\
&F(5) = X(5) f^{(1)}(X(0)) + (X(2)X(3) + X(1)X(4)) f^{(2)}(X(0)) + \frac{1}{2!}(X(1))^2 X(3) +\\
&+X(1)(X(2))^2) f^{(3)}(X(0)) + \frac{1}{3!}(X(1))^3 X(2) f^{(4)}(X(0)) + \frac{1}{5!}(X(1))^5 f^{(5)}(X(0)), \ldots.
\end{aligned} \tag{12}$$

Taking into account that components of differential image of original of non-linear function of differential equation are similar in form to the corresponding components of Adomian polynomials, we can take, that components of differential image of original of non-linear function of equation can be obtained from corresponded components of Adomian polynomials by replacing each solution components $x_k(t)$ by the corresponding component of differential image $X(k)$ of the same index.

In [13] shown, that such replacing can be applied for any types of nonlinearity of differential equations. Therefore, for solution of non-linear boundary value problem we can applied the combined method of differential transformations with approximation of non-linear term of equation by Adomian polynomials [15].

Solution algorithm of non-linear boundary value problem with regard of mentioned above approach consider via example of solving of two-point non-linear boundary value problem, in which object is described by non-linear differential equation:

$$\dot{x}(t) = f[x(t)],\ t \in [t_0, T] \tag{13}$$

with non-linear by x source term $f[x(t)]$ and boundary conditions:

$$\dot{x}(t_0) = 0 \tag{14}$$

$$\alpha \cdot x(T) + \beta \dot{x}(T) = \gamma \tag{15}$$

where α, β, γ are given constants, initial condition $x(t_0)$ is unknown. Is assumed, that function $x(t)$, its derivatives and also non-linear at x function $f[x(t)]$ are continuous functions and equation (13) has unique solution.

Solution algorithm of non-linear boundary value problem (13)-(15) with application of the modified differential transform method consists of realization the following steps [16].

1. Constructing of the spectral model of equation (13) into the image field:

$$(k+1)X(k+1) = F(k), \tag{16}$$

where $F(k)$ is the differential image of non-linear function $f[x(t)]$.

2. Taking into account (1) and boundary condition (14) the value of first discrete of solution is $X(1) = \dot{x}(t_0) = 0.$ For zero discrete assume $X(0) = x(t_0) = \eta$, where the value of parameter η will be determined in further.

3. Differential image $F(k)$ is replaced by corresponding Adomian polynomials, in which each solution component $x_k(t)$ is replaced on corresponding component of differential image $X(k)$ of the same index:

$$F(k) = \widetilde{A}_k,\ k = 0,1,2,\ldots,N. \tag{17}$$

4. Substituting (17) to (16), with regard (3) and step 2 we obtain the solution of non-linear boundary value problem (13) - (15) as:

$$\mathrm{x(t)} = \eta + \sum_{k=1}^{N-1} \frac{\widetilde{A}_{k-1}}{k+1} t^{k+1}. \tag{18}$$

5. Substituting the solution (18) to boundary condition (15) we obtain the non-linear algebraic equation for determination of unknown parameter η

6. Taking into account of founded parameter value and expression (18) we obtain the solution of non-linear boundary value problem (13)-(15).

6. The system-analogue simulation method

Consider solving of non-linear boundary value problem by the system-analogue simulation method [11,17]. Let's, the mathematical model of non-linear boundary value problem is described by vector differential equation:

$$\frac{dx}{dt} = f(t, x) \tag{19}$$

at additional conditions

$$g(x_0, x_n) = p, \tag{20}$$

$$t \in [t_0, t_n], \tag{21}$$

where $x(t)$ is n-measurement of vector-function of state, $f(t,x)$ is n-measurement vector-function, $g(x_0, x_n)$ is n-measurement vector-function of initial and terminal state of object, p is the given vector; $x_0 = x(t_0), x_n = x(t_n)$ are vectors of boundary conditions.

Following [17], carry out the construction of spectral model of nonlinear boundary value problem (19)-(21) as follows. Time interval (21), on which j- component of vector solution of boundary problem should be defined, we cover by the grid with variable step H_j

$$t_i = t_{i-1} + H_{ji}, i = (\overline{1,N}). \tag{22}$$

The interval (21) is usually split into paired quantity of sub-intervals. Around the arbitrary point $t_i \in [t_0, t_n]$ consider separately two sub-intervals: $[t_{i-1}, t_i]$ and $[t_i, t_{i+1}]$. Over the sub-interval $[t_{i-1}, t_i]$ we use the mathematical model of boundary value problem in the form (19)-(20). For the sub-interval $[t_i, t_{i+1}]$ we transform the differential equation (19) by introducing the inverse argument: $\tau = t_{i+1} - t$:

$$\frac{d\bar{x}}{d\tau} = -\bar{f}(\tau, \bar{x}), \tag{23}$$

where the line over symbols means that corresponding variables and functions are reviewed by the inverse argument τ.

Therefore, over the sub-interval $[t_{i-1}, t_i]$ we use the mathematical model (19)-(20) and over the sub-interval $[t_i, t_{i+1}]$ - the inverse model (20), (23).

Applying differential transformation (5) in the point t_{i-1} to the equation (19), obtain the recurrent expression for determination of discretes of differential spectrum over the sub-interval $[t_{i-1}, t_i]$:

$$X_j[k+1, t_{i-1}, x(t_{i-1})] = \\ = \frac{H_{ji}}{k+1} F_j\{T(k, t_{i-1}), X[k, t_{i-1}, x(t_{i-1})]\}, \tag{24}$$

$$j = \overline{1,n},\ i = 2\nu - 1,\ \nu = 1,2,...,N/2,\ k = 0,1,2,...$$

Differential spectrum for the sub-interval $[t_i, t_{i+1}]$ we form by the recurrent expression based on differential transformation (5) of equation (23) in the point t_{i+1} :

$$\bar{X}_j[k+1, t_{i+1}, x(t_{i+1})] = \\ = -\frac{\bar{H}_{ji+1}}{k+1} \bar{F}_j\{\tau(k, t_{i+1}), \bar{X}[k, t_{i+1}, x(t_{i+1})]\}, \tag{25}$$

$$j = \overline{1,n},\ i = 2\nu - 1,\ \nu = 1,2,...,N/2,\ k = 0,1,2,...$$

The function value $x(t)$ in the point $t = t_i$ may obtain by the inverse transformations (6) using differential spectrum (24) or (25). As result of inverse transformations (6) of differential spectra (24) and (25) at conditions $H_{ji} = t_{ji} - t_{ji-1}$, $\bar{H}_{ji+1} = t_{ji+1} - t_{ji}$ obtain the system of equations:

$$X_j(t_{ji}) = \sum_{k=0}^{\infty} X_j[k, t_{i-1}, x(t_{i-1})] = \\ = \sum_{k=0}^{\infty} \bar{X}_j[k, t_{i+1}, x(t_{i+1})], \tag{26}$$

$$j = \overline{1,n}, i = 2\nu - 1, \nu = 1,2,...,N/2$$

The expression (26) we transform as:

$$\sum_{k=0}^{m_1} X_j[k, t_{i-1}, x(t_{i-1})] = \sum_{k=0}^{m_2} \bar{X}_j[k, t_{i+1}, x(t_{i+1})], \tag{27}$$

$$R_{ji} = \bar{R}_{ji},\ j = \overline{1,n},\ i = 2\nu - 1,\ \nu = 1,2,..., \\ n = N/2, \tag{28}$$

where

$$R_{ji} = \sum_{k=m_1+1}^{\infty} X_j[k, t_{i-1}, x(t_{i-1})], \bar{R}_{ji} = \\ = \sum_{k=m_2+1}^{\infty} \bar{X}_j[k, t_{i+1}, x(t_{i+1})].$$

Representation of the system of equations (26) in the form (27) is assumed at condition of equal signs of remainders of a Taylor series R_j and $\bar{R}_j$. System of equations (27) determines on the finite number of discretes of differential spectra (24) and (25) the exact mathematical link between function values $x(t)$ in points t_{i-1} and t_{i+1}. The exact link between $x(t_{i-1})$ and $x(t_{i+1})$ established by the system of equations (27) thanks to joint compensation of remainders R_j and $\bar{R}_j$ of a Taylor series at realization of conditions (28).

Remainders of a Taylor series are expressed exactly in forms of Lagrange and Taylor-Lagrange:

$$R_{ji} = \frac{H_{ji}^{m_1+1}}{(m_1+1)!} x_j^{(m_1+1)}(t_{ji}^*) = \\ = \frac{H_{ji}^{m_1+1}}{m_1!} \int_0^1 (1-\theta)^{m_1} x^{(m_1+1)}(t_{i-1} + \theta H_{ji}) d\theta, \tag{29}$$

$$\bar{R}_{ji} = \frac{\bar{H}_{ji}^{m_2+1}}{(m_2+1)!} x_j^{(m_2+1)}(t_{ji}^{-*}) = \\ = \frac{\bar{H}_{ji}^{m_2+1}}{m_2!} \int_0^1 (1-\bar{\theta})^{m_2} x^{(m_2+1)}(t_{i+1} - \bar{\theta}\bar{H}_{ji}) d\bar{\theta}, \tag{30}$$

where $t_{ji}^* \in [t_{ji-1}, t_{ji}]$; $t_{ji}^{-*} \in [t_{ji}, t_{ji+1}]$.

The system of finite equations (27) – (30) determines the exact link between vector components of boundary conditions and jointly with equation (20) allows to find unknown vectors x_0 and x_N. Expressions (28)-(30) are used for selection of the step H_{ji} and $\overline{H}_{ji}$ by time, the system of equations (20) and (27) determines unknown boundary conditions x_0 and x_N.

The system-analogue model (20), (27) - (30) of non-linear boundary value problem (19) - (21) allows to find exact values of vector components of unknown boundary conditions x_0 and x_N.

7. Numerical experiment

Let's the boundary value problem is given and described by the system of non-linear differential equations [16,17]:

$$\begin{aligned} &\dot{x}_1(t) = x_2, \\ &\dot{x}_2(t) = 1 - x_2^2 \end{aligned} \tag{31}$$

at the additional conditions

$$\begin{aligned} &x_{20} + x_{2T} = th1, \\ &x_1(0) = x_{10} = 0,\ x_1(T) = x_{1T} = \ln(ch1), \\ &x_2(0) = x_{20},\ x_2(T) = x_{2T},\ t \in [0,T], T = 1. \end{aligned} \tag{32}$$

In this example, unknown boundary conditions x_{20}, x_{2T} are connected by the first equation of system (32) and the boundary value problem (31)-(32) has the following exact solution:

$$\begin{aligned} &x_1(t) = \ln(cht),\ x_2(t) = tht, \\ &x_1(0) = 0,\ x_1(T) = \ln(ch1), \\ &x_2(0) = 0,\ x_2(T) = th1. \end{aligned} \tag{33}$$

Consider the solution of the given problem by mentioned above methods.

Solution using the modified differential transform method

Write the boundary value problem (31)-(32) in the image field:

$$\begin{aligned} &X_1(k+1) = \frac{1}{k+1} X_2(k), \\ &X_2(k+1) = \frac{1}{k+1}[\text{ъ}(k) - \tilde{A}_{2k}], \\ &X_2(0) + X_2(T) = th1, \\ &X_1(0) = x_{10} = 0,\ X_1(T) = x_{1T} = \ln(ch1), \\ &X_2(0) = x_{20},\ X_2(T) = x_{2T},\ t \in [0,T], T = 1 \end{aligned} \tag{34}$$

where $\text{ъ}(k) = \begin{cases} 1, & k = 0 \\ 0, & k \geq 1 \end{cases}$.

In accordance with procedure (11) for the non-linear part of the second equation of the system of equations (31) $f_2[x_2(t)] = x_2^2(t)$ we calculate components A_{2k} of Adomian polynomials. Then, we use A_{2k} for definition of corresponding components $\tilde{A}_{2k}$ which are used for replacing of corresponding components of differential image of non-linear part of second equation in the spectral model (34):

$$\begin{aligned} &\tilde{A}_{20} = X_2^2(0),\ \tilde{A}_{21} = 2X_2(0)X_2(1), \\ &\tilde{A}_{22} = X_2^2(1) + 2X_2(0)X_2(2), \\ &\tilde{A}_{23} = 2X_2(0)X_2(3) + 2X_2(1)X_2(2), \\ &\tilde{A}_{24} = 2X_2(0)X_2(4) + 2X_2(1)X_2(3) + X_2^2(2), \\ &\tilde{A}_{25} = 2X_2(0)X_2(5) + 2[X_2(2)X_2(3) + X_2(1)X_2(4)], \ldots \end{aligned} \tag{35}$$

Using the spectral model (34), with taking into account computed values $\tilde{A}_{ik}$ for $k = 0,1,2,\ldots$, obtain following expressions for determination of first 6 discretes of differential spectra, which are expressed through unknown boundary condition x_{20}:

$$\begin{aligned} &X_1(0) = x_{10} = 0,\ X_2(0) = x_{20}, \\ &X_1(1) = x_{20},\ X_2(1) = (1 - x_{20}^2), \\ &X_1(2) = \frac{1}{2}(1 - x_{20}^2),\ X_2(2) = -x_{20}(1 - x_{20}^2), \\ &X_1(3) = -\frac{1}{3}x_{20}(1 - x_{20}^2), \\ &X_2(3) = -\frac{1}{3}\left[1 - 4x_{20}^2 + 3x_{20}^4\right] \\ &X_1(4) = -\frac{1}{12}\left[1 - 4x_{20}^2 + 3x_{20}^4\right], \\ &X_2(4) = \frac{1}{3}\left[2x_{20} - 5x_{20}^3 + 3x_{20}^5\right], \\ &X_1(5) = \frac{1}{15}\left[2x_{20} - 5x_{20}^3 + 3x_{20}^5\right], \\ &X_2(5) = -\frac{1}{15}\left[17x_{20}^2 - 30x_{20}^4 + 15x_{20}^6 - 2\right], \\ &X_1(6) = -\frac{1}{90}\left[17x_{20}^2 - 30x_{20}^4 + 15x_{20}^6 - 2\right], \\ &X_2(6) = -\frac{1}{45}\left[17x_{20} - 77x_{20}^3 + 105x_{20}^5 - 45x_{20}^7\right] \end{aligned} \tag{36}$$

Therefore, taking into account (3), we have the following solution of boundary value problem, which is depended from unknown boundary condition x_{20}

$$
\begin{aligned}
x_1(t) &= x_{20}\cdot t + \frac{1}{2}(1-x_{20}^2)\cdot t^2 - \\
&- \frac{1}{3}x_{20}(1-x_{20}^2)\cdot t^3 - \\
&- \frac{1}{12}\left[1-4x_{20}^2+3x_{20}^4\right]\cdot t^4 + \\
&+ \frac{1}{15}\left[2x_{20}-5x_{20}^3+3x_{20}^5\right]\cdot t^5 - \\
&- \frac{1}{90}\left[17x_{20}^2-30x_{20}^4+15x_{20}^6-2\right]\cdot t^6 - \dots,
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
x_2(t) &= x_{20} + (1-x_{20}^2)\cdot t - x_{20}(1-x_{20}^2)\cdot t^2 - \\
&- \frac{1}{3}\left[1-4x_{20}^2+3x_{20}^4\right]\cdot t^3 + \frac{1}{3}\left[2x_{20}-5x_{20}^3+3x_{20}^5\right]\cdot t^4 - \\
&- \frac{1}{15}\left[17x_{20}^2-30x_{20}^4+15x_{20}^6-2\right]\cdot t^5 - \\
&- \frac{1}{45}\left[17x_{20}-77x_{20}^3+105x_{20}^5-45x_{20}^7\right]\cdot t^6 - \dots .
\end{aligned}
$$

From solving $x_2(t)$ at $t=1$ with taking into account the boundary condition $x_{20}+x_{2T}=th1$ we obtain the equation for determination of unknown boundary condition x_{20}, which solving in the given interval gives the value $x_{20}=0{,}00015$.

Therefore, solving of boundary value problem (31)-(32) which is searching:

$$
\begin{aligned}
x_1(t) &= 0.00015t + 0.5t^2 - 0.00005t^3 - \\
&- 0.08333t^4 + 0.00002t^5 - 3.33\cdot 10^{-6}t^6 - \dots \\
x_2(t) &= 0{,}00015 + t - 0.00015\cdot t^2 - \\
&- 0.33333\cdot t^3 + 0.0001\cdot t^4 - 0.00002\cdot t^5 - \\
&- 0.00005\cdot t^6 - \dots
\end{aligned} \tag{38}
$$

Solution using the system-analogue simulation method [11,16].

The interval $[0,\mathrm{T}]$ is divided on two sub-intervals $\mathrm{H}_1=h$ and $\mathrm{H}_2=1-h$, so that $\mathrm{H}_1+H_2=T=1$. By applying of differential transformations (1) to the system (31) in the point $\mathrm{t}=0$, we obtain the following spectral model:

$$
\begin{aligned}
&X_1(k+1) = \frac{\mathrm{h}}{k+1}X_2(k), \\
&X_2(k+1) = \frac{h}{k+1}[\eth(k) - \sum_{l=0}^{l=k}X_2(k-l)X_2(l)], \\
&X_1(0) = x_{10} = 0,\ X_2(0) = x_{20}, \\
&\eth(k) = \begin{cases} 1, & k=0 \\ 0, & k\ge 1 \end{cases}.
\end{aligned} \tag{39}
$$

The model (39) is of the form of recurrence equations, which allow to find discretes of differential spectra:

$$
\begin{aligned}
&X_1(0)=x_{10}=0,\ X_2(0)=x_{20},\ X_1(1)=\mathrm{h}x_{20},\ X_2(1)=h(1-x_{20}^2), \\
&X_1(2)=\frac{h^2}{2}(1-x_{20}^2), \qquad X_1(3)=-\frac{h^3}{3}x_{20}(1-x_{20}^2), \\
&X_2(2)=-h^2x_{20}(1-x_{20}^2),\ X_2(3)=-\frac{h^3}{3}\left[1-4x_{20}^2+3x_{20}^4\right],
\end{aligned} \tag{40}
$$

$$
\begin{aligned}
&X_1(4) = -\frac{h^4}{12}\left[1-4x_{20}^2+3x_{20}^4\right], \\
&X_2(4) = \frac{h^4}{3}\left[2x_{20}-5x_{20}^3+3x_{20}^5\right], \\
&X_1(5) = \frac{h^5}{15}\left[2x_{20}-5x_{20}^3+3x_{20}^5\right], \\
&X_2(5) = -\frac{h^5}{15}\left[17x_{20}^2-30x_{20}^4+15x_{20}^6-2\right], \\
&X_1(6) = -\frac{h^6}{90}\left[17x_{20}^2-30x_{20}^4+15x_{20}^6-2\right], \\
&X_2(6) = -\frac{h^6}{45}\left[17x_{20}-77x_{20}^3+105x_{20}^5-45x_{20}^7\right].
\end{aligned}
$$

Then, taking into account (3) we have the following solution of boundary value problem (31)-(32) over the sub-interval $\mathrm{H}_1=h$, which is depended from unknown boundary condition x_{20}:

$$
\begin{aligned}
x_1(t) &= hx_{20}\cdot t + \frac{h^2}{2}(1-x_{20}{}^2)\cdot t^2 - \\
&- \frac{h^3}{3}x_{20}(1-x_{20}{}^2)\cdot t^3 - \\
&- \frac{h^4}{12}(1-4x_{20}{}^2+3x_{20}{}^4)\cdot t^4 + \\
&+ \frac{h^5}{15}(2x_{20}-5x_{20}{}^3+3x_{20}{}^5)\cdot t^5 - \\
&- \frac{h^6}{90}(17x_{20}{}^2-30x_{20}{}^4+15x_{20}{}^6-2)\cdot t^6 + \dots,
\end{aligned} \tag{41}
$$

$$x_2(t) = x_{20} + h(1-x_{20}{}^2)\cdot t -$$
$$-h^2 x_{20}(1-x_{20}{}^2)\cdot t^2 - \frac{h^3}{3}(1-x_{20}{}^2)\cdot t^3 -$$
$$-\frac{h^4}{12}(1-4x_{20}{}^2+3x_{20}{}^4)\cdot t^4 +$$
$$+\frac{h^5}{15}(2x_{20}-5x_{20}{}^3+3x_{20}{}^5)\cdot t^5 -$$
$$-\frac{h^6}{90}(17x_{20}{}^2-30x_{20}{}^4+15x_{20}{}^6-2)\cdot t^6+...,$$

Equations (31) transform at the inverse argument $\tau = T - t = 1 - t$:

$$\frac{d\bar{x}_1}{d\tau} = -\bar{x}_2, \quad \frac{d\bar{x}_2}{d\tau} = -1+\bar{x}_2. \tag{42}$$

By applying differential transformations (1) to equations (42) in the point $\tau = 0$ at the scale stationary value $1-h$ we obtain the second spectral model:

$$\bar{X}_1(k+1) = -\frac{1-h}{k+1}\bar{X}_2(k),$$
$$\bar{X}_2(k+1) = \frac{1-h}{k+1}[-\delta(k)+ + \sum_{l=0}^{l=k}\bar{X}_2(k-l)\bar{X}_2(l)], \tag{43}$$
$$\bar{X}_1(0) = \bar{x}_{10} = x_{1T},\ \bar{X}_2(0) = \bar{x}_{20} = x_{2T}.$$

Using recurrence expressions (43) we determine discretes of differential spectra for sub-interval $H_2 = 1-h$:

$$\tilde{X}_1(0) = x_{1T},\ \tilde{X}_2(0) = x_{2T},$$
$$\tilde{X}_1(1) = -(1-h)x_{2T},\ \tilde{X}_2(1) = -(1-h)(1-x_{2T}^2),$$
$$\tilde{X}_1(2) = \frac{(1-h)^2}{2}(1-x_{2T}^2),$$
$$\tilde{X}_2(2) = -(1-h)^2 x_{2T}(1-x_{2T}^2),$$
$$\tilde{X}_1(3) = \frac{(1-h)^3}{3}x_{2T}(1-x_{2T}^2),$$
$$\tilde{X}_2(3) = \frac{(1-h)^3}{3}\left[1-4x_{2T}^2+3x_{2T}^4\right]$$
$$\tilde{X}_1(4) = -\frac{(1-h)^4}{12}\left[1-4x_{2T}^2+3x_{2T}^4\right],$$
$$\tilde{X}_2(4) = \frac{(1-h)^4}{3}\left[2x_{2T}-5x_{2T}^3+3x_{2T}^5\right], \tag{44}$$
$$\tilde{X}_1(5) = -\frac{(1-h)^5}{15}\left[2x_{2T}-5x_{2T}^3+3x_{2T}^5\right],$$
$$\tilde{X}_2(5) = \frac{(1-h)^5}{15}\left[17x_{2T}^2-30x_{2T}^4+15x_{2T}^6-2\right]$$
$$\tilde{X}_1(6) = -\frac{(1-h)^6}{90}\left[17x_{2T}^2-30x_{2T}^4+15x_{2T}^6-2\right],$$
$$\tilde{X}_2(6) = \frac{(1-h)^6}{45}\left[17x_{20}-77x_{20}^3+105x_{20}^5-45x_{20}^7\right]$$

Substituting received discretes to the expression (27), obtain:

$$x_{20} + h(1-x_{20}^2) - x_{20}h^2(1-x_{20}^2) - \frac{h^3}{3}\left[1-4x_{20}^2+3x_{20}^4\right] + \frac{h^4}{3}\left[2x_{20}-5x_{20}^3+3x_{20}^5\right] -$$
$$-\frac{h^5}{15}\left[17x_{20}^2-30x_{20}^4+15x_{20}^6-2\right] - \frac{h^6}{45}\left[17x_{20}-77x_{20}^3+105x_{20}^5-45x_{20}^7\right] =$$
$$= x_{2T} - (1-h)(1-x_{20}^2) - x_{20}(1-h)^2(1-x_{20}^2) + \frac{(1-h)^3}{3}\left[1-4x_{20}^2+3x_{20}^4\right] + \tag{45}$$
$$+\frac{(1-h)^4}{3}\left[2x_{20}-5x_{20}^3+3x_{20}^5\right] + \frac{(1-h)^5}{15}\left[17x_{20}^2-30x_{20}^4+15x_{20}^6-2\right] +$$
$$+\frac{(1-h)^6}{45}\left[17x_{20}-77x_{20}^3+105x_{20}^5-45x_{20}^7\right]$$

The equation (45) with selected the scale stationary value h, jointly with boundary condition $x_{20}+x_{2T} = th1$ creates the system of two equations with two unknowns x_{20}, x_{2T}. Solution of this system gives values of unknown boundary conditions in the given interval: $x_{20} = 0{,}000331;\ x_{2T} = 0{,}761263.$

Substituting obtained values to expressions (41) we receive the solution of boundary value problem (31) - (32) as:

$$x_1(t) = 0{,}00033 \cdot t + 0{,}5 \cdot t^2 - 0{,}00011 \cdot t^3 - - 0{,}08333 \cdot t^4 + 0{,}00004 \cdot t^5 + 0{,}02222 \cdot t^6 + ...,$$

$$x_2(t) = 0{,}00033 + 0{,}99999 \cdot t - 0{,}00033 \cdot t^2 - - 0{,}33333 \cdot t^3 + 0{,}00022 \cdot t^4 + 0{,}13333 \cdot t^5 - - 0{,}00013 \cdot t^6 + ...$$

In figure 1 and tables 1-2 shown the comparison between exact solution of boundary value problem (31) - (32), solution using the modified differential transform method and solution using the method of system-analogue simulation, the relative errors for the given solution, which was obtained with using first 6 discretes of differential spectra of the solution is also presented.

Table 1

Comparison of numerical solutions

t	Exact solution		SSM		MDTM	
	$x_1(t)$	$x_2(t)$	$x_1(t)$	$x_2(t)$	$x_1(t)$	$x_2(t)$
0	0	0	0	0.000331	0	0.000149
0.2	0.019868	0.197375	0.019933	0.197694	0.019896	0.197477
0.4	0.077954	0.379949	0.078084	0.380315	0.077923	0.378794
0.6	0.170135	0.537049	0.170415	0.538602	0.169280	0.528105
0.8	0.290754	0.664037	0.291915	0.673199	0.285966	0.629408
1.0	0.433781	0.761594	0.439152	0.800092	0.416783	0.666697

Table 2

Comparison of relative errors

t	Relative error, ε_r			
	SSM		MDTM	
	$x_1(t)$	$x_2(t)$	$x_1(t)$	$x_2(t)$
0	0	4.35e-04	0	1.96e-04
0.2	1.51-04	4.18e-04	6.48e-05	1.34e-04
0.4	2.99e-04	4.80e-04	6.91e-05	1.52e-03
0.6	6.44e-04	2.04e-03	1.97e-03	1.17e-02
0.8	2.68e-03	1.20e-02	1.10e-02	4.55e-02
1.0	1.24e-02	5.05e-02	3.92e-02	1.25e-01

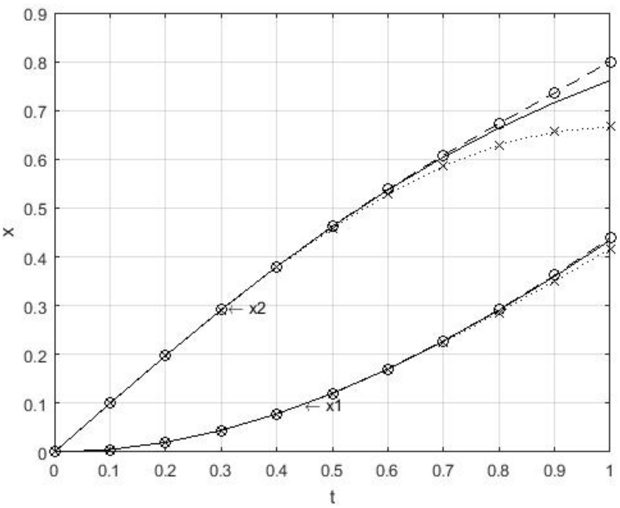

Fig.1. Comparison of exact solution (-) and solutions using SSM (-o-) and MDTM (-x-), which are obtained with using of first 6 discretes

As mentioned in [17], the solution accuracy of boundary value problem by system-analogue simulation method can be raised, if necessary, by raising of amount of considered discretes of differential spectra or by dividing of the given interval on larger number of sub-intervals.

8. Conclusion

The application of two methods, based on differential transformations for solving non-linear boundary value problems was reviewed. Both methods give the good agreement with exact solution of boundary value problem over small intervals. The system-analogue simulation method is more complicated for realization in comparison with modified differential transform method due to necessary to construct additional spectral models (analogs) which leads to increase of dimension of the system of equation. But it application is more preferable in cases of big intervals on which the boundary value problem is considered.

Possible with the purpose of computational cost reduction to apply in the system-analogue simulation method of approximation of non-linear term of differential equations by Adomian polynomials. Effectiveness evaluation of this method is the subject of separate investigations.

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В.П. Гусинін[1], А.В. Гусинін[2], О.М. Тачиніна[3]

Застосування диференціальних перетворень до розв'язку нелінійних крайових задач

[1] Бінаціональна компанія «Alc?ntara Cyclone Space» Av. L3 Norte – Ed. Finatec – Bloco H, Brasilia-DF – Brasil

[2] Національний Технічний Університет України «КПІ» ім. І. Сикорського пр. Перемоги, 37, Київ, 03056, Україна

[3] Національний авіаційний університет, просп. Космонавта Комарова, 1, Київ, Україна, 03680

E-mails: [1]viacheslav.gusynin@gmail.com; [2]gusynin@gmail.com; [3]tachinina@mail.ru

Мета: Метою даної статті є порівняння застосування методів на основі диференціальних перетворень для розв'язку крайових задач, що описуються нелінійними звичайними диференціальними рівняннями. **Методи:** В статті розглянуто два підходи із застосуванням диференціальних перетворень до розв'язку нелінійної крайової задачі: модифікований метод диференціальних перетворень і метод системоаналогового моделювання. **Результати:** Представлені результати численного розв'язку нелінійної крайової задачі методами на основі диференціальних перетворень для демонстрації ефективності та застосовності методів. Наведена відносна похибка для даних розв'язків, отриманих з використанням перших шести дискрет диференціальних спектрів. **Обговорення:** Порівняння чисельних розв'язків, отриманих модифікованим методом диференціальних перетворень і методом системоаналогового моделювання із точним розв'язком показало, що обидва методи мають добру збіжність з точним розв'язком нелінійної крайової задачі на малих інтервалах. При цьому, застосування метода системоаналогового моделювання є більш раціональним на великих інтервалах, на яких розглядається крайова задача.

Ключові слова: диференціальні перетворення; метод системоаналогового моделювання; модифікований метод диференціальних перетворень; нелінійна крайова задача; поліноми Адоміана.

В.П. Гусынин[1], А.В. Гусынин[2], Е.Н. Тачинина[3]. Применение дифференциальных преобразований к решению нелинейных краевых задач

[1] Бинациональная компания «Alc?ntara Cyclone Space» Av. L3 Norte – Ed. Finatec – Bloco H, Brasilia-DF – Brasil
[2] Национальный Технический Университет Украины «КПИ» им. И. Сикорского, пр. Победы, 37, Киев, Украина, 03056,
[3] Национальный авиационный университет, просп. Космонавта Комарова, 1, Киев, Украина, 03680
E-mails: [1]viacheslav.gusynin@gmail.com; [2]gusynin@gmail.com; [3]tachinina@mail.ru

Цель: Целью данной статьи является сравнение применения методов на основе дифференциальных преобразований для решения краевых задач, описываемых нелинейными обыкновенными дифференциальными уравнениями. **Методы:** В статье рассмотрено два подхода с применением дифференциальных преобразований к решению нелинейной краевой задачи: модифицированный метод дифференциальных преобразований и метод системоаналогового моделирования. **Результаты:** Представлены результаты численного решения нелинейной краевой задачи методами на основе дифференциальных преобразований для демонстрации эффективности и применимости методов. Приведена относительная погрешность для данных решений, полученных с использованием первых шести дискрет дифференциальных спектров. **Обсуждение:** Сравнение численных решений, полученных модифицированным методом дифференциальных преобразований и методом системоаналогового моделирования с точным решением показало, что оба метода имеют хорошую сходимость с точным решением нелинейной краевой задачи на малых интервалах. При этом, применение метода системоаналогового моделирования является предпочтительным на больших интервалах, на котором рассматривается краевая задача.

Ключевые слова: дифференциальные преобразования; метод системоаналогового моделирования; модифицированный метод дифференциальных преобразований; нелинейная краевая задача; полиномы Адомиана.

Gusynin Viacheslav. Doctor of Engineering
Acting General Director Ukraine, Binational company «Alc?ntara Cyclone Space»
Education: Taras Shevchenko National University of Kyiv, Kyiv, Ukraine (1973).
Research area: flight dynamics, optimal control, differential transformation.
Publications: 210
E-mail: viacheslav.gusynin@gmail.com

Gusynin Andrii. Candidate of Engineering, Associate Professor.
Associate Professor, National Technical University of Ukraine «Igor Sikorsky KPI»
Education: International Solomon University, Kyiv, Ukraine (2000).
Research area: flight dynamics, automatization, optimal control, differential transformation.
Publications: 105
E-mail: gusynin@gmail.com

Tachinina Helen. Candidate of Engineering, Associate Professor.
Department of Automation and Energy Management, National Aviation University
Education: National Aviation University, Kyiv, Ukraine (2000).
Research area: flight dynamics, automatization, optimal control.
Publications: 100.
E-mail: tachinina@rambler.ru