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K.V.Aasianov, G.N. Bojarsky

THE ESTIMATE MULTIPLE AS A RESULT OF PRIORI LINKS ACCOUNTING. GENERALIZATION OF CRAMER-RAO INEQUALITY

It is shown, that the lowest Cramer-Rao bound for statistical estimation could be changed to accounting a priori links between estimated parameters and the residual variances could be essentially reduced. The case of two parameters being linked with algebraic linear equation to be estimated is considered. In the second part of the paper the results of "concordance" of flight information on the base of use of a centre motion and kinematics Euler relationships as a priori links are represented.

In some important cases, when several parameters have to be estimated the priori known analytical links between them can be used to improve the quality of estimates.

In the rigid body dynamics, an aircraft, for example, the Euler kinematic equations and other kinematic rations can be taken as a priori links. In [1; 2] it was shown, that under some conditions the accounting of the analytical priori links essentially reduces the residual variances of estimations. These variances turn out to be lower than the Cramer-Rao bound, when it is counted under assumption of disregard of a priori links.

We will denote by t the statistics constructed on sample $\{x_1, x_2, \dots x_n\}$ and assume it to be an estimation of the function $\tau(\Theta)$ of the parameter Θ. Then, the Cramer-Rao inequality is

$$D(t) \geq \tau'(\Theta)^2 \left\{ M\left[\left(\frac{\partial \log L}{\partial \Theta}\right)^2\right]\right\}^{-1} \tag{1}$$

where, $D(t)$ denotes that the variance of t, $L(x, \Theta)$ is the likelihood function, M is the symbol of the mean value.

The next properties of the likelihood function will be used:

$$\int \dots \int L dx_1 \dots dx_n = 0;$$

$$\int \dots \int \frac{\partial L}{\partial \Theta} dx_1 \dots dx_n = 0;$$

$$M\left(\frac{\partial \log L}{\partial \Theta}\right) = \int \dots \int \left(\frac{1}{L}\frac{\partial L}{\partial \Theta}\right) L dx_1 \dots dx_n = 0. \tag{2}$$

$$\int \dots \int \left[\left(\frac{1}{L}\frac{dL}{d\Theta}\right)^2 - \frac{d^2 \log L}{d\Theta^2}\right] L dx_1 \dots dx_n = 0 .$$

When τ is unbiased estimate of $\tau(\Theta)$, there exists the following equality:

$$\int \dots \int t L dx_1 \dots dx_n = \tau(\Theta)$$

and therefore

$$\int\ldots\int \frac{\partial \log L}{\partial \Theta} L dx_1 \ldots dx_n = \int\ldots\int (t - \tau(\Theta)) \frac{\partial \log L}{\partial \Theta} L dx_1 \ldots dx_n = \tau'(\Theta) ,$$

where

$$\tau'(\Theta) = \frac{\partial \tau(\Theta)}{\partial \Theta} .$$

The inequalities (1) turn into equality if and only if the proportionality between $t - \tau(\Theta)$ and $\partial \log L / \partial \Theta$ exists, and the coefficient of proportionality is function of Θ only:

$$\frac{\partial \log L}{\partial \Theta} = A(\Theta)(t - \tau(\Theta)) .$$

Then, the variance $D(t)$ can be expressed as

$$D(t) = \tau'(\Theta)[A(\Theta)]^{-1} .$$

We will consider the next task. The estimates of parameters Θ_1 and Θ_2 have to be obtained, when likelihood function is $L = L(x_1, \ldots x_n | \Theta_1, \Theta_2)$. According to the maximum likelihood method the estimates of Θ_1 and Θ_2 can be found as a solution of equations

$$\frac{\partial \log L}{\partial \Theta_1} = 0; \ \frac{\partial \log L}{\partial \Theta_2} = 0 .$$

Assume, that the analytical link between parameters Θ_1 and Θ_2 exists in the next form;

$$G(\Theta_1, \Theta_2) = 0 .$$

This equation is determined (not stochastic). In order to find the estimates of Θ_1 and Θ_2 the Lagrange function Φ can be composed as

$$\Phi^* = \log L + \lambda G .$$

The estimates of Θ_1 and Θ_1 will be found as a solution of equations

$$\frac{\partial \log L}{\partial \Theta_1} + \lambda \frac{\partial G}{\partial \Theta_1} = 0 ; \ \frac{\partial \log L}{\partial \Theta_2} + \lambda \frac{\partial G}{\partial \Theta_2} = 0 .$$

The consideration bellow refers to the case of function $G(\Theta_1, \Theta_2, \ldots, \Theta_n)$ that condition $M[\lambda] = 0$ should be fulfilled.

Let us introduce the magnitudes containing unknown parameters α_{ij}:

$$\bar{D}_1 = t_1 - \tau_1(\Theta_1) - \alpha_{11}\left(\frac{\partial \log L}{\partial \Theta_1} + \lambda \frac{\partial G}{\partial \Theta_1}\right) - \alpha_{12}\left(\frac{\partial \log L}{\partial \Theta_2} + \lambda \frac{\partial G}{\partial \Theta_2}\right) ;$$

$$\bar{D}_2 = t_2 - \tau_2(\Theta_2) - \alpha_{21}\left(\frac{\partial \log L}{\partial \Theta_1} + \lambda \frac{\partial G}{\partial \Theta_1}\right) - \alpha_{22}\left(\frac{\partial \log L}{\partial \Theta_2} + \lambda \frac{\partial G}{\partial \Theta_2}\right) .$$

Mean values of $\overline{D}_1^2$ and $\overline{D}_2^2$ are expressed as following:

$$M\left(\overline{D}_1^2\right)=\int\ldots\int\left[t_1-\tau_2(\Theta_2)-\alpha_{11}\left(\frac{\partial\log L}{\partial\Theta_1}+\lambda\frac{\partial G}{\partial\Theta_1}\right)-\alpha_{12}\left(\frac{\partial\log L}{\partial\Theta_2}+\lambda\frac{\partial G}{\partial\Theta_2}\right)\right]^2\cdot Ldx_1\ldots dx_n\,; \quad (3)$$

$$M\left(\overline{D}_2^2\right)=\int\ldots\int\left[t_2-\tau_2(\Theta_2)-\alpha_{21}\left(\frac{\partial\log L}{\partial\Theta_1}+\lambda\frac{\partial G}{\partial\Theta_1}\right)-\alpha_{22}\left(\frac{\partial\log L}{\partial\Theta_2}+\lambda\frac{\partial G}{\partial\Theta_2}\right)\right]^2\cdot Ldx_1\ldots dx_n\,. \quad (4)$$

The determination of elements of matrix $\alpha=\|\alpha_{ij}\|(ij\in 1,2)$ can be done on the way of minimising magnitudes $M\left(\overline{D}_1^2\right)$ and $M\left(\overline{D}_2^2\right)$ with respect to α_{ij}. That leads to equations

$$\frac{\partial M\left(D_1^2\right)}{\partial\alpha_{11}}=0;\quad \frac{\partial M\left(\overline{D}_1^2\right)}{\partial\alpha_{12}}=0;\quad \frac{\partial M\left(D_2^2\right)}{\partial\alpha_{21}}=0;\quad \frac{\partial M\left(D_1^2\right)}{\partial\alpha_{22}}=0. \quad (5)$$

Introduce next designations:

$$\mu_1=\frac{\partial\log L}{\partial\theta_1}+\lambda\frac{\partial G}{\partial\theta_1};\quad \mu_2=\frac{\partial\log L}{\partial\theta_2}+\lambda\frac{\partial G}{\partial\theta_2};$$

$$M_{ij}=\int\ldots\int\mu_i\mu_j Ldx_1\ldots dx_n,\qquad (i\in 1,2);$$

$$R_{ij}=\int\ldots\int t_i\lambda\frac{\partial G}{\partial\theta_j}Ldx_1\ldots dx_n\,.$$

Then, in accordance with (5), the system for determination of α_{ij} after corresponding with transformation can be written as

$$\begin{bmatrix} M_{11} & M_{12} & 0 & 0\\ M_{12} & M_{22} & 0 & 0\\ 0 & 0 & M_{11} & M_{12}\\ 0 & 0 & M_{12} & M_{22}\end{bmatrix}\begin{bmatrix}\alpha_{11}\\ \alpha_{12}\\ \alpha_{21}\\ \alpha_{22}\end{bmatrix}=\begin{bmatrix}\tau_1'(\theta_1)+R_{11}\\ R_{12}\\ R_{21}\\ \tau_2'(\theta_1)+R_{11}\end{bmatrix}$$

Thus, the coefficients α_{ij} are expressed in the following forms

$$\alpha_{11}=\frac{\left(\tau_1^1(\theta_1)+R_{11}\right)M_{22}-R_{12}M_{12}}{\Delta};$$

$$\alpha_{12}=\frac{M_{11}R_{12}-\left(\tau_1^1(\theta_1)+R_{11}\right)M_{12}}{\Delta};$$

$$\alpha_{21}=\frac{M_{22}R_{21}-\left(\tau_2^1(\theta_2)+R_{22}\right)M_{12}}{\Delta};$$

$$\alpha_{22}=\frac{\left(\tau_2^1(\theta_2)+R_{22}\right)M_{11}-R_{21}M_{12}}{\Delta},$$

where $\Delta=M_{11}M_{22}-M_{12}^2$.

Let us denote $D_i(t_i) = \int ... \int (t_i - \tau_i(\theta_i))^2 L dx_1 ... dx_n$, where $D_i(t_i)$ is the variance of statistics t_i that is estimate of function $\tau_i(\theta_i)$.

Taking into account this determination of $D(t_i)$ and substituting the values of α_{ij} given above in the right-hand parts of (3) and (4) we can find inequalities

$$D_1(t_1) \geq \frac{1}{\Delta}\left[\tau_1'^2(\theta_1)M_{22} - 2\tau_1'(\theta_1)(R_{12}M_{12} - R_{11}M_{22}) + R_{11}^2 M_{22} + R_{12}^2 M_{11} - 2R_{11}R_{22}M_{12}\right]; \tag{6}$$

$$D_2(t_2) \geq \frac{1}{\Delta}\left[\tau_2'^2(\theta_2)M_{11} - 2\tau_2'(\theta_2)(R_{21}M_{12} - R_{22}M_{11}) + R_{21}^2 M_{22} + R_{22}^2 M_{11} - 2R_{21}R_{22}M_{12}\right]. \tag{7}$$

These inequalities are the results of positiveness of values $M(D_2')$. They generalize in some sense is the Cramer-Rao inequalities, and tend to them, when λ turns to zero. If estimates are biased then

$$M(t_i) = \int ... \int t_i L dx_1 ... dx_n = T_i(\theta) = \tau_i(\theta) + \delta_i(\theta)$$

where $\delta_i(\theta)$ are biases. The corresponding inequalities for variances would be obtained from (6) and (7) with substitution $T_i'(\theta)$ instead of $\tau_i(\theta)$.

In the case when $\tau_i(\theta_i) = \theta_i$ we can see that $\tau_i'(\theta) = 1 + \delta_i'(\theta_i)$.

The many – dimensionality of vector – parameter $\theta = (\theta_1, \theta_2, ..., \theta_n)$ estimation and existence of $k < m$ priori links will not bring principle difference in the course of calculation.

A simple example will be considered assuming that $G = \Theta_1 + \Theta_2 - a = 0$ and, in addition, that

$$\left(\frac{\partial \log L}{\partial \Theta_1}\right)^2 = \frac{n_1^2}{\sigma_1^4}(\bar{x}_1 - \Theta_1)^2 ; \left(\frac{\partial \log L}{\partial \Theta_2}\right)^2 = \frac{n_1^2}{\sigma_2^4}(\bar{x}_2 - \Theta_2)^2 .$$

Then,

$$\left(\frac{\partial \log L}{\partial \Theta_1}\right)^2 = \left(\frac{\partial \log L}{\partial \Theta_2}\right)^2 = 1 .$$

In previous formulas

$$\bar{x}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} x_{ij} .$$

For the sake of simplification the stochastic values $(x_1 - \Theta_1)$ and $(x_2 - \Theta_2)$ are assumed to be uncorrelated. This corresponds to the case when parameters Θ_1 and Θ_2 are being estimated and the probability distribution is

$$dF\left(x_{1_j}, x_{2_k}\right) = \frac{1}{2\pi\sigma_1\sigma_2}\exp\left\{-\frac{1}{2}\frac{\left(x_{1_j} - \Theta_1\right)^2}{\sigma_1^2} - \frac{1}{2}\frac{\left(x_{2_k} - \Theta_2\right)^2}{\sigma_2^2}\right\} dx_{1_j} dx_{2_k} .$$

Presuppose that

$$M\left[\left(\frac{\partial \log L}{\partial \Theta_1}\right)^2\right] = \frac{n_1}{\sigma_1^2}\,; \qquad M\left[\left(\frac{\partial \log L}{\partial \Theta_2}\right)^2\right] = \frac{n_2}{\sigma_2^2}$$

and

$$M\left[\frac{\partial \log L}{\partial \Theta_1}\,\frac{\partial \log L}{\partial \Theta_2}\right] = 0\,.$$

In the considered case the Lagrange coefficient can be simply obtained

$$\lambda = \frac{n_1 n_2}{n_2\sigma_1^2 + n_1\sigma_2^2}\left(d - \bar{x}_1 - \bar{x}_2\right)$$

where n_1 and n_2 denote different sample values. Then the estimates of θ_1 and θ_2 are not $\bar{x}_1$ and $\bar{x}_2$ as in the separated estimations, but

$$\hat{\theta}_1 = \left(\sigma_1^2 n_2 + \sigma_2^2 n_1\right)^{-1}\left[\sigma_2^2 n_1 \bar{x}_1 + \sigma_1^2 n_2 (d - \bar{x}_2)\right]$$

$$\hat{\theta}_2 = \left(\sigma_1^2 n_2 + \sigma_2^2 n_1\right)^{-1}\left[\sigma_1^2 n_2 \bar{x}_2 + \sigma_2^4 n_1 (d - \bar{x}_1)\right]$$

If we denote D_i^0 the variance of separated estimates (without account of the link), the variances of improved estimates with account of the link are

$$D\left(\hat{\theta}_1\right) = D_1^0\left(1 + \frac{\sigma_1^2 n_2}{\sigma_2^2 n_1}\right)^{-1} = D_1^0 K_1$$

$$D\left(\hat{\theta}_2\right) = D_2^0\left(1 + \frac{\sigma_2^2 n_1}{\sigma_1^2 n_2}\right)^{-1} = D_2^0 K_2$$

It is seen that coefficient K_i belongs to the range $[0, 1]$. So $D\left(\hat{\theta}_i\right) \le D_i^0$.

Assume that $n_1 = n_2$, $\sigma_1 = \sigma_2$, then $K_1 = K_2 = 0{,}5$ and $D\left(\hat{\theta}_i\right) = 0{,}5 D_i^0$.

It is also seen that if measurements of x_2 are absolutely precise: $\sigma_2 = 0$ but $\sigma_1 \neq 0$, the variance of estimate $\hat{\theta}_1$ appears nevertheless to be equal zero.

Because of $D_1^0 = \frac{\sigma_1^2}{n_1}$; $D_2^0 = \frac{\sigma_2^2}{n_2}$ it could be found, that

$$D\left(\hat{\theta}_1\right) = D\left(\hat{\theta}_2\right) = \frac{\sigma_1^2\sigma_2^2}{n_1\sigma_2^2 + n_2\sigma_1^2} = \frac{D_1^0 D_2^0}{D_1^0 + D_2^0}\,.$$

The obtained inequalities can be used to estimate the effectiveness of the accounting of the priori information and the corresponding complication of algorithms that appear in that case in different applied tasks.

In solving the problem of obtaining flight parameter estimates we can use as a priori links the equations of motion of plane centre mass expressed in load factors, kinematics Euler relationships, kinematics relationships, defining position of plane centre of mass with regard to ground system of co-ordinates, geometric relationships between angles, between angles and velocity projections. All these relationships are non-linear and much more complicated in comparison with a priori link recorded in the first part of this article.

The method of parameter co-ordination was elaborated by G.N. Boyarsky, and A.V. Godovanuke [3].

Let $\bar{y}(t_j), t_j \in [t_0, t_n]; j \in \overline{1, j}$ be an array of measured data. The set of equations mentioned above which have to be used as the priori links is the following

$$\dot{x} = f(x,t); \; \psi(x,t) = 0; \; y = h(x), \tag{8}$$

where $x - n$ is dimension vector, f –and $\Psi - n_1$ is dimension and n_2 is dimension functions respectively, $y - k$ is dimension vector of observable parameters.

Suppose that it is possible in some way to estimate the phase co-ordinates $\hat{x}(t)$, expressed in terms of estimates of observable parameters $\hat{y}(t) = y(t) = \Delta\hat{y}(t)$. Then estimate $\hat{x}(t)$ could be used to calculate secondary estimate of observable parameters

$$\tilde{y}(t) = h(\hat{x}(t)).$$

Co-ordination of data array is considered to be complete if the following conditions are satisfied

$$|\tilde{y}_m - \hat{y}(t)_m| \le \varepsilon_m ; \hat{y}_m(t) \le= h_m ; \; \forall t \in [t_0, t_n].$$

Where h_m is normalised precision of y_m parameter measurement, $\varepsilon_m < h_m$ is a given accuracy of co-ordination problem solution.

It has been shown that co-ordination as a problem can be reformulated as a minimisation problem of discrepancies $\tilde{y}(t)$ and $\hat{y}(t)$:

$$\Delta\hat{y}^*(t) = \arg \min_{\Delta\hat{y} \in \Omega} Q,$$

$$Q = \sum_{m=1}^{k} \left\{ \frac{\mu_m}{h^2 (t_k - t_0)^2} \int_{t_0}^{t_k} [\tilde{y}_m(t) - \hat{y}(t)]^2 dt + d_m \max_{t \in [t_0, t_k]} |\Delta\hat{y}_m(t)| \right\}.$$

Where μ_m are coefficients, expressing degree of confidence to y_m array quality weight coefficients d_m satisfy inequality $d_m << \mu_m [h(t_k - t_0)]^{-2}$.

The second inequality system in (8) is a set of restrictions in conditional optimisation problem, the first one – its completion characteristic. Vector $\tilde{y}(t)$ is taken as co-ordinated data array.

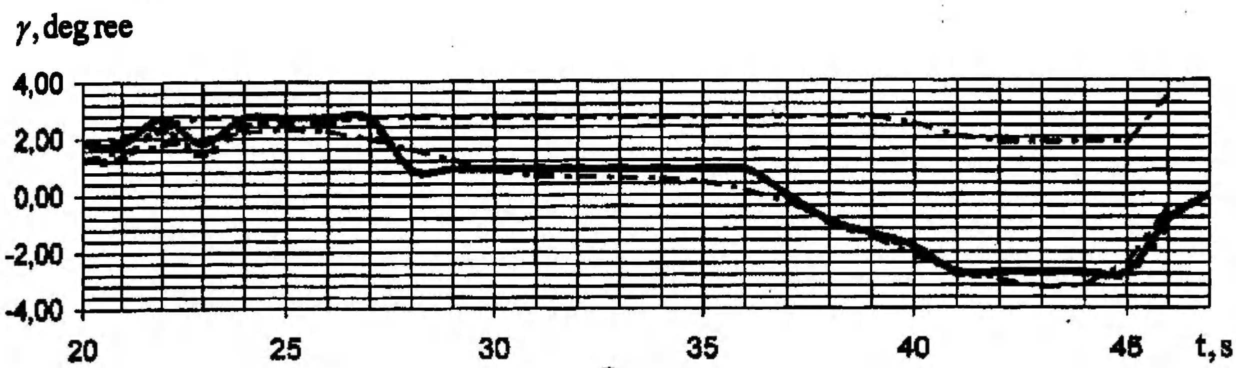
Estimate model of measurement error vector $\hat{y}(t)$ is chosen in wave form $\hat{y}(t) = A\varphi(t)$, where A is matrix of unknown random parameters, being constant on every i-interval of time $t \in [t_{i-1}, t_i]$, $\varphi(t)$ is a vector of basis functions – Chebyshev polynomials. From the random

A process point of view expression for $\Delta y(t)$ for fixed matrix A can be considered as an analytical representation of multiparameter parent population containing the best estimation. It is worth indicating that $\Delta y(t)$ may be not ergodic function.

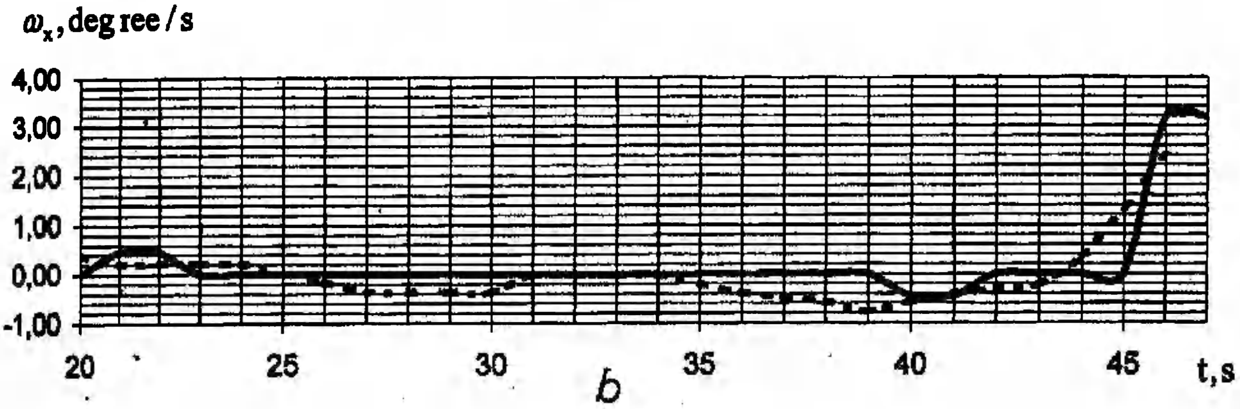
This shortly described approach of estimate procedure design with use a priori links is wider than the one mentioned above in the first part of this paper because of possibility to take into account systematic errors.

In conclusion we will show some results of flight data processing with use of algorithm of co-ordination method applied to estimating the trajectory parameters related to the landing accident of AN-225 plane in "Ramenskoe" airport. Results of calculation are represented on Fig.1 – 7. Analysis of these results reveals low quality of initial data and necessity to apply flight parameter co-ordination procedure.

Fig.1 shows light degree of readability of estimate of trajectory parameters related to the crew radiocommunication. It is seen that while turning to runway the plane touched the ground by wing tip and the sixth engine pod. The described method can be used for the estimation of flight parameters in the incident (accident) investigation.

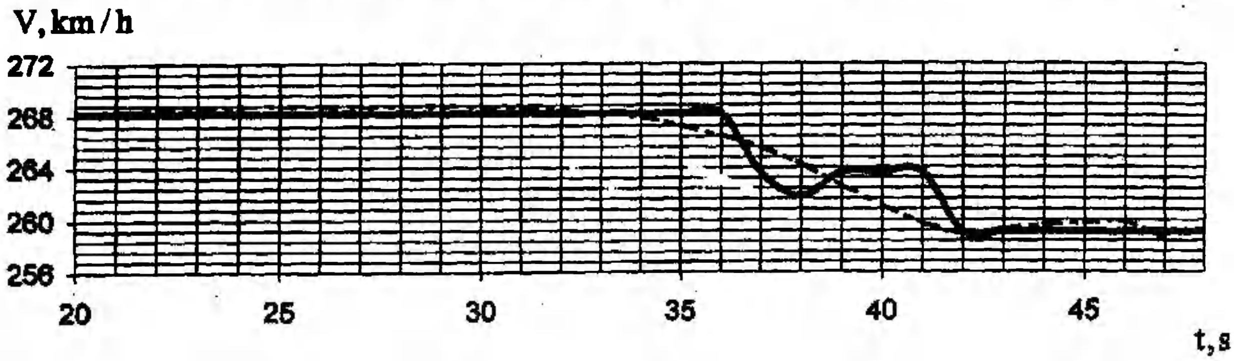


a

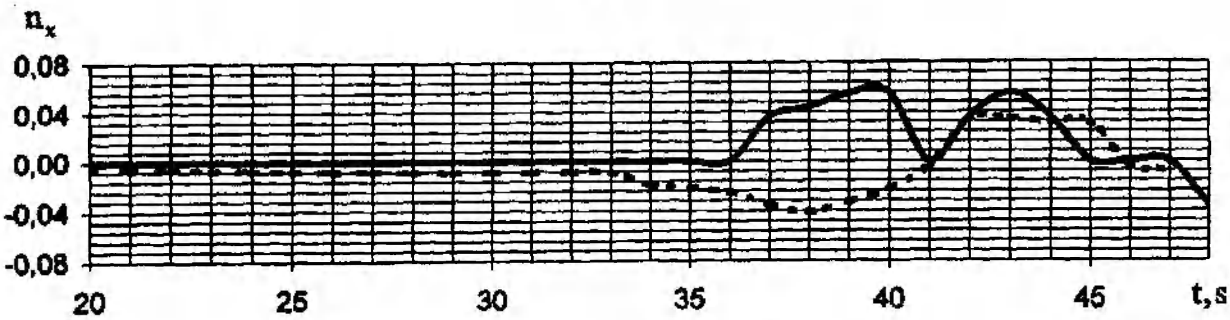


b

Fig. 1. Roll angle and roll angular estimated with the help of Euler kinematic relationships:
a-roll angle; b-roll angular velocity; ——— recording; – – – – – – primary estimate; –·–·–·–·· secondary estimate; –··–··–· noncorrected data estimate



a



b

Fig. 2. Plane flight speed on landing approach phase and estimate of longitudinal load factor: a-flight speed; b-longitudinal load factor; ———— recording; - - - - - - - estimate

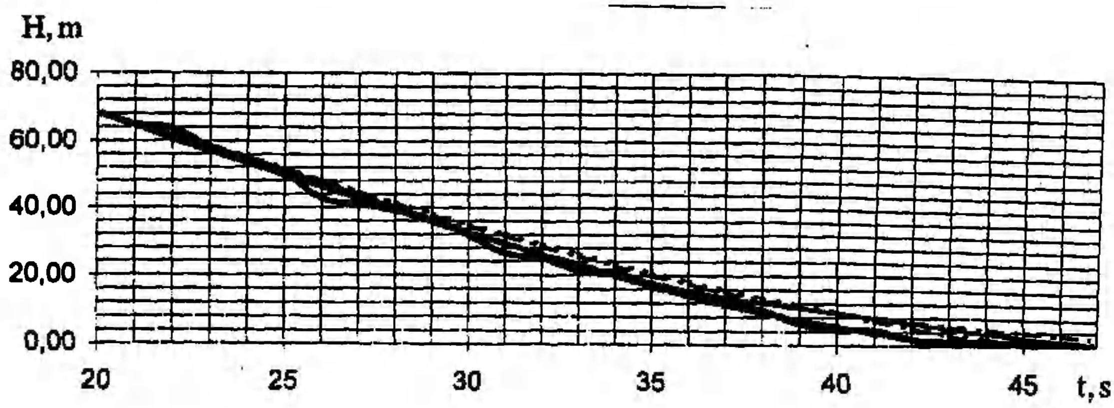


Fig.3. Plane flight altitude estimate on landing approach phase:
——— recording; ——— J3=0,704; ——— J3=0,895 ;
········ J3=1,118; - · - · - J3=1,162; - · · - · · J3=1,191

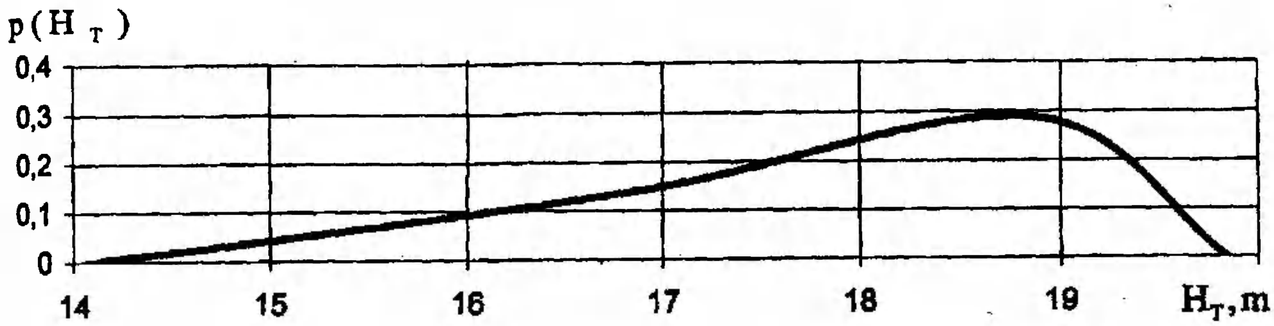


Fig. 4. Statistical estimate of density distribution of ranway hold flyby altitube ;
$M[H_т] = 17{,}7$ м; $\sigma_{H_т} = 1{,}5$ м

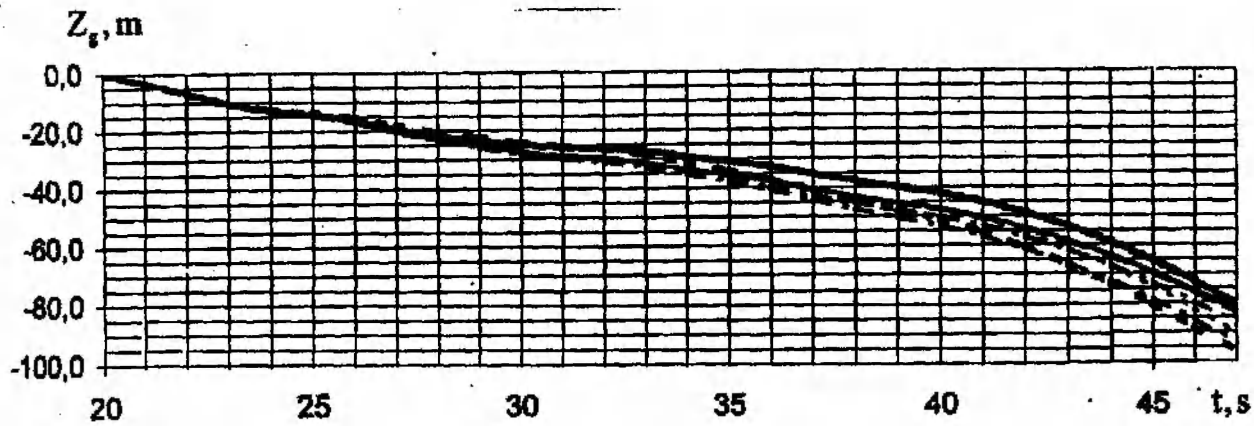


Fig. 5. Plane lateral deviation on landing approach phase:
J3=0,704; J3=0,895; J3=1,089:
J3=1,118; J3=1,162; J3=1,191; J3=2,841

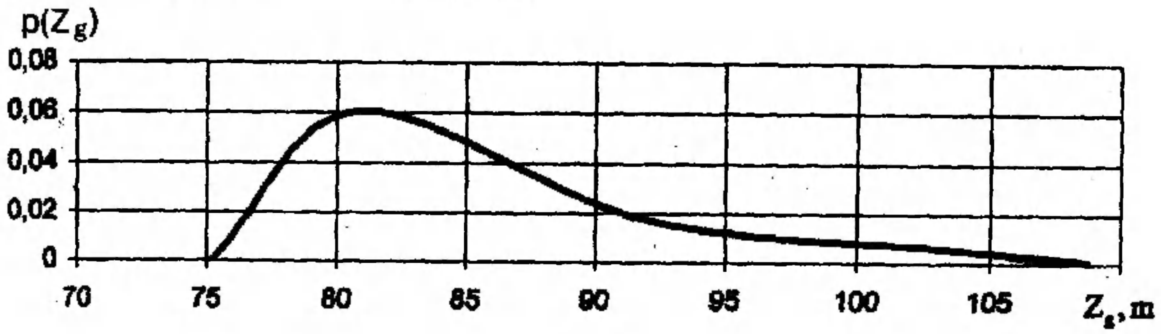


Fig. 6. Statistical estimate of density distribution funxtion of plane maximum lateral deviation on landing approach phase:
$M\lfloor Z_g \rfloor = 84{,}8m$; $\sigma_{z_g} = 8{,}7m$

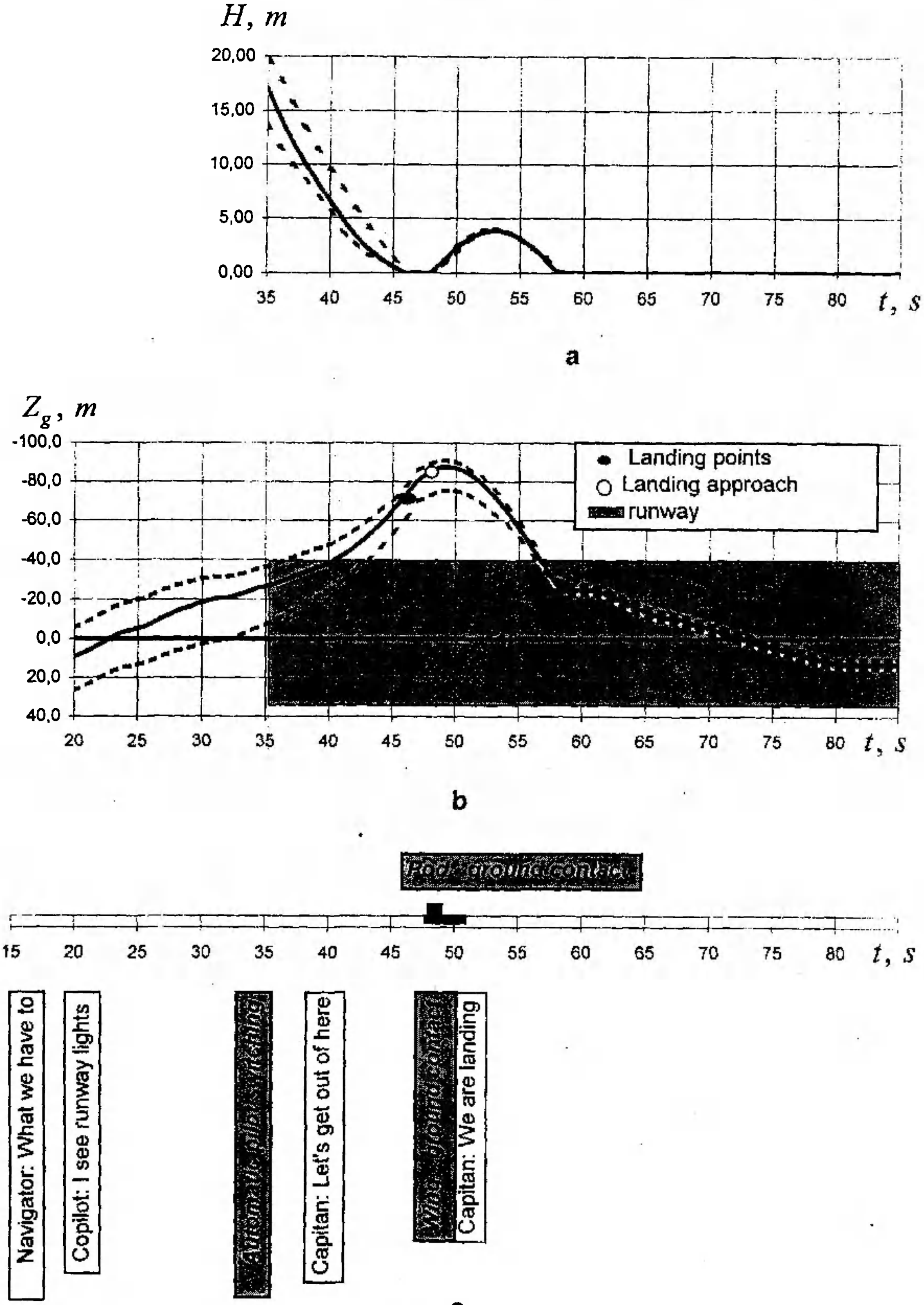


Fig. 7. Plane flight profile estimate:
a-altitude; b-lateral deviation; c-pilot reporting

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Володимир Олександрович Кас'янов (1935) закінчив Миколаївський судобудівельний інститут в 1959 році. Доктор технічних наук професор завідувач кафедри теоретичної механіки Київського міжнародного університету цивільної авіації, член експертної ради ВАК України та спеціалізованих рад з захисту дисертацій. Автор понад 170 публікацій з різних областей: аеромеханіка, динаміка, безпека польотів, керування динамічними системами, ідентифікація, прикладна статистика та обробка експериментальних даних, деякі проблеми мікроекономіки.

Vladimir A. Kasianov (b. 1935) graduated from Nikolayev Shipbuilding Institute (1959). DSc (Eng) professor, head of Theoretical Mechanics Department of Kyiv International University of Civil Aviation. Author of more than 170 publications in the sphere of aeromechanics, electrodynamics, statical mechanics, flight dynamics, safety of flight, dynamical system control, identification theory, applied statistics, experimental data processing, some problems of microeconomics. Member of Expert Council of HAC of Ukraine, member of Specialized Councils for thesis defense.

Георгій Миколайович Боярський (1935-1998) закінчив Миколаївський судобудівельний інститут в 1959 році. Кандидат технічних наук доцент. Автор понад 50 публікацій з різних областей: електрогазодинаміка, методи та алгоритми статистичного моделювання польоту, оптимального керування щодо задач формування льотної придатності, розслідування льотних випадків, вдосконалення методів льотної експлуатації.

Georgy N. Bojarsky (1935-1998) graduated from Nikolayv Shipbuilding Institute (1959). PhD (Eng) ass. professor. Author of more than 50 publication in electrodynamics, methods and algorithms of statistical simulation of flight optimum control applicated to the problems of air worthiness, flight accidents investigations, flight operation methods improving.