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GREEN'S FUNCTION OF THE CONVECTIVE WAVE EQUATION FOR A RIGID RECTANGULAR PIPE

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Green's function of the convective wave equation for an infinite straight rigid pipe of rectangular cross-section is found. This function is written in terms of series of the pipe acoustic modes. Each term of the series is a sum of the direct and back waves propagating in the corresponding mode downstream and upstream, respectively, of the unit point impulse acoustic source. In the found Green's function, the uniform mean flow effects are reflected in the direct form. In obtaining this function, the combinations of appropriate mathematical operations are found, which allow one to reduce the one-dimensional convective Klein-Gordon equation to its classical one-dimensional counterpart, and, on the basis of the known solution of the later equation, obtain the solution of the former one.

Keywords: convective wave equation; Green's function; rigid rectangular pipe.

Побудовано функцію Гріна конвективного хвильового рівняння для нескінченної прямої жорсткої труби прямокутного поперечного перерізу. Ця функція записується у вигляді ряду по акустичних модах труби. Кожен член ряду є сумою прямої та зворотної хвиль, які поширюються на відповідній моді вниз та вгору за течією від одиничного точкового імпульсного акустичного джерела. У знайденій функції Гріна в явному вигляді відображені ефекти рівномірної середньої течії. При її побудові знайдено комбінації відповідних математичних операцій, котрі дозволяють зводити одновимірне конвективне рівняння Кляйна-Гордона до його класичного одновимірного аналогу, і, на основі відомого розв'язку останнього рівняння, знаходити розв'язок першого.

Ключові слова: конвективне хвильове рівняння; функція Гріна; жорстка прямокутна труба.

Introduction

Problems of finding and studying acoustic fields in pipes of different geometries and sizes are of a great concern in car- and aircraft-building industry, gas and oil industry, municipal economy, architecture, medicine, etc. [1–4]. Independently of the pipe type and the acoustic sources in pipes, all of these problems can be, in principle, solved by the Green's function technique. However, the application of this technique is only reasonable when the principal possibility of finding the Green's function of interest exists.

Apart from the professional qualification and skill of the investigator, this possibility depends on many factors. These include the geometry of the pipe under investigation and the shape of its cross-section, the physical properties of the pipe wall and the type of its support, the physical properties of the internal and external media, the acoustical conditions at the pipe ends, presence or absence of internal flow in the pipe, etc.

As analysis of the scientific literature shows, among the cases, which are specified by various combinations of the noted factors, the most investigated are the cases of an infinite straight rigid pipe of circular and rectangular cross-section [5–8]. For these cases, the corresponding Green's functions of the wave and Helmholtz equations have been found, and, based on the Green's function technique, expressions for the various characteristics of the acoustic fields generated by sources of interest in the noted pipes have been obtained. However, usually all of these results are restricted to the case of flow absence in the pipe. When the inner flow in pipe is taken into consideration, its effects in the corresponding Green's functions and/or the final results are only reflected in the indirect form[1] [1, 5–8].

This disadvantage is partially corrected in the present paper. Here the Green's function of the three-dimensional wave equation for an infinite straight immovable rigid pipe of rectangular cross-

[1] In the direct form (i.e., in the form of direct mathematical dependencies of the investigated acoustic field characteristics on the flow parameters), these effects are only reflected in the appropriate scaling laws and/or various quantitative estimates.

section with inner uniform mean flow is found. The found function has the direct dependence on the flow parameters, and, in the case of flow absence, coincides with the corresponding Green's function for the investigated pipe, which is available in the scientific literature.

Formulation of the problem

An infinite straight immovable rigid-walled pipe of rectangular cross-section of dimensions l_x and l_y is considered, in which a fluid[2] flows uniformly with the mean axial velocity U. In this pipe, the acoustic sources of any nature are given, which are distributed in the arbitrary manner and generate sound. The generated sound field is governed by the specific type of the three-dimensional wave equation, which is often called the three-dimensional *convective* wave equation[3] [7, 8], viz.

$$\frac{1}{c_0^2}\frac{d^2 p_a}{dt^2} - \nabla^2 p_a = \gamma \tag{1}$$

(here p_a is the acoustic pressure; c_0 the sound speed in the undisturbed fluid; t the time, and γ the function describing the total distribution of the noted sources). It is necessary to find the Green's function of equation (1) for the pipe under consideration.

Green's function

The Green's function, $\mathrm{G}(\mathbf{r},t;\mathbf{r}_0,t_0)$, to be found satisfies the following equation

$$\frac{1}{c_0^2}\frac{d^2 G}{dt^2} - \nabla^2 G = \delta(r - r_0)\delta(t - t_0) \tag{2}$$

(in which $\delta(\mathbf{r}-\mathbf{r}_0)$ and $\delta(t-t_0)$ are the spatial three-dimensional and temporal one-dimensional Dirac delta-functions, respectively), and describes the acoustic pressure at the field point $\mathbf{r}$ at the time t, which is generated in the pipe at the moment t_0 by a point impulse acoustic source of the unit amplitude located at the point $\mathbf{r}_0$.

In the rectangular Cartesian coordinate system, (x, y, z), which is employed for solving the formulated problem, equation (2) has the following form

$$\frac{1}{c_0^2}\frac{d^2 G}{dt^2} - \frac{\partial^2 G}{\partial x^2} - \frac{\partial^2 G}{\partial y^2} - \frac{\partial^2 G}{\partial z^2} = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0)\delta(t - t_0). \tag{3}$$

Here the second total temporal derivative is written in the following way

$$\frac{d^2}{dt^2} = \left(\frac{\partial}{\partial t} + \mathbf{U}\cdot\nabla\right)^2 = \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial z}\right)^2 = \frac{\partial^2}{\partial t^2} + 2U\frac{\partial^2}{\partial t \partial z} + U^2\frac{\partial^2}{\partial z^2}; \tag{4}$$

the mean flow velocity vector, $\mathbf{U}$, and the nabla-operator, ∇, look as

$$\mathbf{U} = U_x\mathbf{e}_x + U_y\mathbf{e}_y + U_z\mathbf{e}_z = U\mathbf{e}_z ;$$

$$\nabla = \frac{\partial}{\partial x}\mathbf{e}_x + \frac{\partial}{\partial y}\mathbf{e}_y + \frac{\partial}{\partial z}\mathbf{e}_z ;$$

the dot between the vectors $\mathbf{U}$ and ∇ indicates their scalar product; $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$ are the unit directivity vectors of the axes x, y and z, respectively; the axis z is directed along the flow; the beginning of the coordinate system (x, y, z) is taken at arbitrary position of any pipe rib; and the variables in equation (3) vary in the following ranges:

$$0 \le x, x_0 \le l_x ; \quad 0 \le y, y_0 \le l_y ; \quad |z| < \infty ; \quad |z_0| < \infty ;$$
$$|t| < \infty ; \quad |t_0| < \infty .$$

The boundary conditions for the function G are that its normal derivative is equal to zero on the immovable rigid pipe wall, viz.

$$\left.\frac{\partial \mathrm{G}}{\partial x}\right|_{x=0,l_x} = 0 , \quad \left.\frac{\partial \mathrm{G}}{\partial y}\right|_{y=0,l_y} = 0 , \tag{5}$$

and that all waves are outgoing at infinity. Apart from these, G should also satisfy the causality condition [5–8], viz.

$$\left.\mathrm{G}\right|_{t<t_0} = 0 . \tag{6}$$

The first of them indicates that the normal components of the acoustic velocity vanish on the wall, the second one that there is no sound reflection at the pipe ends (at infinity), whereas the third one means that there is no acoustic field in the pipe before the beginning of sound generation by the source.

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[2] Neither the fluid viscosity nor its mass density is considered in this study. It is explained by the fact that, in the problem formulated in such a manner, the first fluid characteristic will play no role at all (because the generated sound is considered to propagate in the inviscid compressible fluid [1, 5–8]), whereas the second one will be only reflected in the final result in the indirect form (i.e., via the given sound speed in the undisturbed medium, c_0).

[3] The presence of the term "convective" in the name of this equation is due to that it has the total temporal derivative, $\mathrm{d}/\mathrm{d}t = \partial/\partial t + \mathbf{U}\cdot\nabla$, which has the non-zero convective derivative, $\mathbf{U}\cdot\nabla$, caused by the uniform mean flow in the pipe (here $\mathbf{U}$ is the mean flow velocity vector, ∇ the gradient, and the dot between these vectors indicates their scalar product) [7, 8]. In the case of the flow absence (i.e., $\mathbf{U} = 0$) the convective derivative in equation (1) vanishes, and it coincides with its classical three-dimensional counterpart.

The solution to the problem (3)–(6) is sought in the form of expansion of the function G into an infinite series of the pipe acoustic modes, $Ш_{nm}(x,y)=\cos(k_{xn}x)\cos(k_{ym}y)$, viz.

$$G(x,y,z,t;x_0,y_0,z_0,t_0)= =\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} G_{nm}(z,t;x_0,y_0,z_0,t_0)\,\Psi_{nm}(x,y), \qquad (7)$$

where $k_{xn}=n\pi/l_x$ and $k_{ym}=m\pi/l_y$ are the modal wavenumbers in the x- and y- directions, respectively. This representation of the Green's function satisfies condition (5) identically.

In series (7), the unknown are the coefficients G_{nm}. In order to find them, expression (7) is substituted into equation (3) (rewritten with the use of (4)), after that the obtained relationship is multiplied scalarly by the functions Ψ_{nm}, and the orthogonality property of Ψ_{nm} is taken account of, viz.

$$\int_0^{l_x}\int_0^{l_y}\Psi_{nm}(x,y)\,\Psi_{sq}(x,y)\,dxdy= =\begin{cases}\|\Psi_{nm}\|^2;(s,q)=(n,m);\\ 0;(s,q)\neq(n,m);\end{cases}$$

$$\|\Psi_{nm}\|^2=\begin{cases} l_x l_y; n=0, m=0;\\ l_x l_y/2; n=0, m\ge 1;\\ l_x l_y/2; n\ge 1, m=0;\\ l_x l_y/4; n\ge 1, m\ge 1.\end{cases} \qquad (8)$$

This results in the equation for G_{nm}, viz.

$$\frac{1}{c_0^2}\frac{\partial^2 G_{nm}}{\partial t^2}+2\frac{M}{c_0}\frac{\partial^2 G_{nm}}{\partial t\partial z}- -\left(1-M^2\right)\frac{\partial^2 G_{nm}}{\partial z^2}+k_{nm}^2 G_{nm}= =\frac{\Psi_{nm}(x_0,y_0)}{\|\Psi_{nm}\|^2}\delta(z-z_0)\delta(t-t_0), \qquad (9)$$

in which $M=U/c_0$ is the uniform mean flow Mach number in the pipe, $k_{nm}^2=k_{xn}^2+k_{ym}^2$ the modal wavenumbers in the plane xy, the magnitudes $\|\Psi_{nm}\|^2$ are given in (8), and the variation ranges of the variables x_0, y_0, z, z_0, t and t_0 are presented before (5).

The analysis of equation (9) shows that, except for the terms containing the number M, it coincides with the classical one-dimensional Klein-Gordon equation[4], whose solution for an infinite domain is well known [6, 7]. In order to get rid of these terms and, hence, proceed from (9) to the noted equation, let us introduce the following non-dimensional variables

$$Z=\lambda\frac{z}{l},\ Z_0=\lambda\frac{z_0}{l},\ T=\lambda^{-1}\frac{c_0 t}{l}+M\lambda\frac{z}{l},$$
$$T_0=\lambda^{-1}\frac{c_0 t_0}{l}+M\lambda\frac{z_0}{l},\ \lambda=\frac{1}{\sqrt{1-M^2}} \qquad (10)$$

in which the length scale, l, can be chosen in the arbitrary manner (the corresponding arguments are given after relationship (13) (see there footnote 5)).

In the variables (10), the convective terms[4] in the left part of equation (9) vanish, and it becomes the above-noted Klein-Gordon equation [6, 7], viz.

$$\frac{\partial^2 G_{nm}}{\partial T^2}-\frac{\partial^2 G_{nm}}{\partial Z^2}+k_{nm}^2 l^2 G_{nm}= =l^2\frac{\Psi_{nm}(x_0,y_0)}{\|\Psi_{nm}\|^2}\delta\left(\frac{l}{\lambda}(Z-Z_0)\right)\delta\times \times\left(\frac{\lambda l}{c_0}(T-T_0-M(Z-Z_0))\right) \qquad (11)$$

in the domain $|Z|<\infty$, $|Z_0|<\infty$, $|T|<\infty$, $|T_0|<\infty$.

The solution to equation (11) in the indicated domain is a superposition of the direct and back waves propagating to the right and to the left, respectively, from the impulse source located at the point $Z=Z_0$ [6, 7], viz.

$$G_{nm}=\frac{c_0}{2}\frac{\Psi_{nm}(x_0,y_0)}{\|\Psi_{nm}\|^2}\left[H(Z_0-Z)\times H(T-T_0+Z-Z_0)+ +H(Z-Z_0)H(T-T_0-(Z-Z_0))\right]\times \times J_0\left(k_{nm}l\sqrt{(T-T_0)^2-(Z-Z_0)^2}\right). \qquad (12)$$

Here

$$H(x)=\int_{-\infty}^{x}\delta(\eta)d\eta=\begin{cases}0, x<0\\ 1, x\ge 0\end{cases}$$

is the Heaviside unit-step function [6–8], and the radiation-at-infinity condition for the function G (see immediately after (5)) has been taken account of.

Taking account of transformation (10) in formula (12) yields the final expressions for the functional coefficients G_{nm} in series (7), viz.[5]

[4] Since the noted terms appear in (9) due to the convective derivative, $U\partial/\partial z$, in equation (3), they can be called the convective ones, and equation (9) – the one-dimensional convective Klein-Gordon equation.

[5] One can see that actually none of the functions in relationship (13) depends on the length scale l (because

$$G_{nm}=\frac{c_0}{2}\frac{\Psi_{nm}(x_0,y_0)}{\|\Psi_{nm}\|^2}\times$$
$$\times\left[H\left(\frac{\lambda}{l}(z_0-z)\right)H\left(\frac{c_0}{\lambda l}(t-t_0)+(M+1)\frac{\lambda}{l}(z-z_0)\right)+\right.$$
$$\left.+H\left(\frac{\lambda}{l}(z-z_0)\right)H\left(\frac{c_0}{\lambda l}(t-t_0)+(M-1)\frac{\lambda}{l}(z-z_0)\right)\right]\times$$
$$\times J_0\left(k_{nm}l\sqrt{\frac{c_0^2}{\lambda^2l^2}(t-t_0)^2+2\frac{c_0M}{l^2}(t-t_0)\times}\right.$$
$$\left.\times\sqrt{(z-z_0)+(M^2-1)\frac{\lambda^2}{l^2}(z-z_0)^2}\right), \tag{13}$$

where J_0 is the cylindrical Bessel function of zeroth order.

Then substituting magnitudes (13) into relationship (7) allows one to find the expression for the required Green's function of equation (1) for the pipe under consideration, viz.

$$\mathrm{G}(x,y,z,t;x_0,y_0,z_0,t_0)=$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\mathrm{G}_{nm}(z,t;x_0,y_0,z_0,t_0)\Psi_{nm}(x,y)=$$
$$=\frac{c_0}{2}\left[H\left(\frac{\lambda}{l}(z_0-z)\right)H\left(\frac{c_0}{\lambda l}(t-t_0)+(M+1)\frac{\lambda}{l}(z-z_0)+\right)\right.$$
$$+H\left(\frac{\lambda}{l}(z-z_0)\right)\times$$
$$\left.\times H\left(\frac{c_0}{\lambda l}(t-t_0)+(M-1)\frac{\lambda}{l}(z-z_0)\right)\right]\times$$
$$\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\Psi_{nm}(x_0,y_0)}{\|\Psi_{nm}\|^2}\Psi_{nm}(x,y)\times$$
$$\times J_0\left(k_{nm}l\sqrt{\frac{c_0^2}{\lambda^2l^2}(t-t_0)^2+2\frac{c_0M}{l^2}(t-t_0)(z-z_0)}\right.$$
$$\left.\sqrt{+(M^2-1)\frac{\lambda^2}{l^2}(z-z_0)^2}\right), \tag{14}$$

in which the variation ranges of all the arguments of the function G are given before (5).

One can see that the obtained Green's function (14) is written in terms of series of the pipe acoustic modes, Ψ_{nm}. Each term of the series is a sum of the direct and back waves propagating in the corresponding mode downstream and upstream, respectively, of the unit point impulse acoustic source located in the pipe cross-section $z=z_0$. Apart from this, as it should be, the function G satisfies the causality condition (6), as well as conditions (5) and the radiation-at-infinity condition (see above).

Further analysis of relationship (14) shows that in the found Green's function, the mean flow effects are reflected in the direct form (via the numbers M and $\lambda=\lambda(M)$). The effects become more significant as the flow Mach number, M, increases, causing, in particular, the appearance and further growth of the function asymmetry about the plane $z=z_0$ in which the noted source is located. And vice versa, the decrease of the Mach number results in the decrease of the effects and, in particular, the decrease of the indicated asymmetry. In the case of mean flow absence (i.e., $M=0$, $\lambda=1$) the function (14) becomes symmetric about the plane $z=z_0$ and coincides with the Green's function of the classical three-dimensional wave equation for the investigated pipe, which is available in the scientific literature [1, 6, 7], viz.

$$G\big|_{M=0}=\frac{c_0}{2}\left[H\left(\frac{1}{l}(z_0-z)\right)H\left(\frac{c_0}{l}(t-t_0)+\frac{1}{l}(z-z_0)\right)+\right.$$
$$\left.+H\left(\frac{1}{l}(z-z_0)\right)H\left(\frac{c_0}{l}(t-t_0)-\frac{1}{l}(z-z_0)\right)\right]\times$$
$$\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\Psi_{nm}(x_0,y_0)}{\|\Psi_{nm}\|^2}\Psi_{nm}(x,y)\times$$
$$J_0\left(k_{nm}\sqrt{c_0^2(t-t_0)^2-(z-z_0)^2}\right).$$

Conclusion

1. The Green's function of the three-dimensional wave equation for an infinite straight immovable rigid pipe of rectangular cross-section with uniform mean flow has been obtained in terms of the pipe acoustic modes.

2. In this function, each term of the series is a sum of the direct and back waves propagating in the corresponding pipe acoustic mode downstream and upstream, respectively, of the unit point impulse acoustic source.

3. In the found Green's function, the mean flow effects are reflected in the direct form. The effects become more significant as the flow Mach number increases, causing, in particular, the appearance and further growth of the functions asymmetry about the pipe cross-section $z=z_0$ in which the above-noted acoustic source is located. And vice versa, the decrease of the Mach number results in the decrease

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the functions Ψ_{nm} and J_0 are independent of l at all, whereas the scale $l>0$ does not influence the signs of the arguments of the Heaviside functions). This indicates that l can be chosen in the arbitrary manner in transformation (10). For example, it can be equal to either l_x or l_y or $\sqrt{(l_x/2)^2+(l_y/2)^2}$, etc.).

of the effects and, in particular, the decrease of the indicated asymmetry.

4. In the case of uniform mean flow absence the obtained Green's function is symmetric about the section $z = z_0$ and coincides with the corresponding Green's function for the investigated pipe, which is available in the scientific literature.

5. In obtaining the Green's function, the combinations of appropriate mathematical operations have been found, which allow one to reduce the one-dimensional convective Klein-Gordon equation (9) to its classical one-dimensional counterpart (11), and, on the basis of the known solution of the later equation, obtain the solution of the former one.

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