UDC 532.526.72 DOI: 10.18372/0370-2197.3(104).18981

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THE EFFECT OF PARTIAL SLIP ON THE SURFACE PRESSURE DISTRIBUTION IN A SLIGHTLY COMPRESIBLE FLOW DEVELOPMENT REGION IN THE BOUNDARY LAYER

The study of laminar incompressible fluid flow in the boundary layer revealed, even earlier, that the condition of complete adhesion of fluid particles to the surface (non-slip condition) of the moving body (half-plane) is not met in the flow development (formation) region. The assumption of constancy of the fluid velocity on the surface of a moving body, hence non-slip, leads, in the flow development region, to the complete absence of the normal component of the velocity field. And this contradicts the very concept of the flow development region, where there should be two velocity components - longitudinal (primary) and normal (secondary) ones. In the previous works of the authors, analytical solutions were obtained for the velocity field in the region of development of incompressible fluid flow in the boundary layer. Since the use of the incompressible fluid flow model is restricted by the Mach number, to further expand the speed range, the problem of the of slightly compressible fluid flow development region in the boundary layer was considered. It is analytically proven that all considerations regarding the impossibility of complete non-slip in the flow development region can be applied to a slightly compressible flow. Slight compressibility at the same time means the subsonic nature of the flow and the neglect of temperature effects due to friction. On the basis of a critical analysis of the existing approaches, which consider the flow of a fluid around a immobile plate in the framework of non-gradient flow (which is just impossible due to the lack of a mechanism for creating the motion of the fluid), it is shown that the system of equations is actually non-closed. For the region of flow development, where the longitudinal pressure gradient is not a constant value, one equation is missing. This equation, as in previous works, is obtained from the necessary condition for the extreme of the fluid rate functional. And although the complete solution for the longitudinal component of the velocity contains four constants of integration, to obtain the asymptotics near the solid surface it is sufficient to know only two quantities - the velocity and its first derivative (gradient). These values, as it turns out from the asymptotic solution, coincide with the case of incompressible flow, which allows us to expand the scope of the previously obtained results for a wider domain of Mach numbers, for example $\mathrm{Ma} \le (0.5-0.6)$. *And such values already correspond to the speeds of modern civil aircraft. The dimensionless distribution of pressure in the slightly compressible flow development region is presented and its significant heterogeneity is shown, which, in turn, indicates the importance of the obtained results.*

Keywords: *slightly compressible flow, flow development region, boundary layer, pressure distribution, fatigue stresses and surface deformation, Navier-Stokes equations*

Introduction. In modern aviation, the speeds of airplanes and helicopters exceed the limit of 0.2-0.3 Ma, and the flow of fluid (air) caused by friction between the surface of the aircraft and the air should be considered compressible. We will show that, as in the case of an incompressible flow [1], all conclusions remain valid for a slightly compressible flow. So, consider an infinite plate moving at a constant velocity directed along the plane of this plate [1]. A boundary layer is formed in the region of flow development, in which there are two velocity components: longitudinal and transverse. At the same time, it is essential that by the end of the flow development region (in dimensionless coordinates, see [1])

$$V_y \to 0,\ x \to -1.$$

Since the discovery of the phenomenon of partial slippage [1] is based on the use of the continuity equation (conservation of mass), let us consider this equation for the case of a compressible flow as well. We have

$$\frac{\partial(\rho V_x)}{\partial x} + \frac{\partial(\rho V_y)}{\partial y} = 0,$$

or, in expanded form

$$\frac{\partial \rho}{\partial x} V_x + \rho \frac{\partial V_x}{\partial x} + \frac{\partial \rho}{\partial y} V_y + \rho \frac{\partial V_y}{\partial y} = 0. \quad (1)$$

In (1) V_x, V_y, ρ stand, respectively, for longitudinal and transverse velocities, density.

Let's assume, as everyone does, that fluid particles (in this case, air) instantly stick to the surface of a moving half-plane (or body). Then, for all points of the half-plane, the so-called non-slip condition is met

$$V_{x|y=0} = \text{Const=V}_0,\quad V_{y|y=0} = 0. \quad (2)$$

It immediately follows from condition (2):

$$\frac{\partial V_x}{\partial x}_{|y=0} = 0. \quad (3)$$

Equation (1) on the surface of the half-plane simplifies to:

$$\left[\frac{\partial \rho}{\partial x} V_x + \rho \frac{\partial V_y}{\partial y}\right]_{y=0} = 0. \quad (4)$$

Condition (2) also implies the condition of constancy of pressure on the surface of the half-plane:

$$p_{|y=0} = \text{Const}. \quad (5)$$

As a result, the constancy of the density follows from relation (5).

$$\rho_{|y=0} = \text{Const}. \tag{6}$$

It turns out that the continuity equation on the surface of the half-plane is equivalent to this one

$$\left[\rho \frac{\partial V_y}{\partial y}\right]_{y=0} = 0. \tag{7}$$

Due to the finite value of the density, it turns out that (7) is equivalent

$$\frac{\partial V_y}{\partial y}_{|y=0} = 0. \tag{8}$$

From relations (2) and (8), as well as on the basis of the Stokes model of a viscous flow which does not allow the formulation of a boundary condition

$$\frac{\partial^2 V_y}{\partial y^2}_{|y=0} \neq 0,$$

we come to an unambiguous conclusion:

$$V_y \equiv 0. \tag{9}$$

The identity sign instead of equality in (9) means that the normal component of the velocity can be, according to the Stokes theory, only equal to zero everywhere. It is obvious that identity (9) does not hold in the flow development region. And if so, then we can conclude that in the flow development region:

$$\frac{\partial V_x}{\partial x}_{|y=0} \neq 0,\ \frac{\partial p}{\partial x}_{|y=0} \neq 0. \Rightarrow V_x = V_x(x), \quad \rho = \rho(x),\ V_y = 0,\ \frac{\partial V_y}{\partial y} \neq 0; \quad -1 \leq x \leq 0. \tag{10}$$

Allow me, opponents will say, but what about the problem of fluid flowing around the surface of semi-infinite stationary plate? That the non-slip condition is not met there? The thing is that at the initial moments of time when the flow is created (the wind tunnel is turned on), the flow development region has a finite size. But, as it reaches the stationary mode (within hundredths of a second), the flow development region decreases and, ultimately, shrinks to a point at half-plane. And before the beginning of the half-plane, a stagnant zone is formed. Nothing similar, from a physical point of view, happens with a body moving in a stationary fluid: at each subsequent moment of time, new and new fluid particles are involved in the motion and are accelerated from a state of rest to a finite speed in a fraction of a second. There can be no instantaneous acceleration, because there is nowhere to get an infinitely large force.

Problem state. Until recently, it was believed that in the boundary layer of an incompressible fluid flow, molecular viscosity is a constant value that does not depend on spatial coordinates, but can only be a function of temperature. Studies of the laminar boundary layer revealed interesting features. The problem of the motion of an infinite plane was solved only after assuming the variability in space of the molecular diffusion of a fluid [2]. At the same time, the longitudinal velocity distributions in the gradient and non-gradient boundary layers have, as it turned out, completely different functional dependencies: in the non-gradient boundary layer, it is an exponential decrease down to zero [3], while in the gradient boundary layer, it is the well-known parabolic distribution [4-9]. Stokes [10] and Rayleigh [11] obtained a solution to the problem of boosting of a plane with subsequent constant speed of motion, according to which friction stresses disappear after the boosting of the plane. Assumptions of variability, both in space and in time, of molecular diffusion made it possible to obtain a physical solution to the problem, according to which the frictional stress reaches its asymptotics immediately after the boosting is stopped [12]. Since infinite bodies do not exist, the so-called Blasius problem was considered [1]. As it turned out (and it was shown above in this work also for a slightly compressible fluid flow), in the flow development region, it is impossible to meet the non-slip boundary condition exactly, because this is equivalent, within the framework of the Navier-Stokes equations, to the absence of the second component of the velocity. Instead of the non-slip condition in the flow development region, it is necessary to use the partial slip condition [1].

Problem formulation. Extend the concept of the of flow development region to the class of slightly compressible flows, when the compressibility of the flow cannot be neglected, but the effects associated with heating due to the friction of the fluid against the solid surface are still insignificant.

The purpose of the work. The purpose of this work is to obtain, based on an analytical approach, an estimate of the field of the longitudinal component of the velocity near the surface of the moving half-plane. Based on this estimate, obtain the asymptotic distribution of the pressure field near the surface of the half-plane in the slightly compressible flow development region and point to the importance of this distribution both for the lifting force of the wing and for the calculation of stresses arising on the surface of the wing.

Asymptotics of the velocity field in the slightly compressible flow development region in the boundary layer. Consider a plane compressible flow on the surface of an infinite plane. This flow is described by the following dimensionless equations (formulae (51) paragraph 144 [13])

$$\rho V_x \frac{\partial V_x}{\partial x} + \rho V_y \frac{\partial V_x}{\partial y} = -\frac{1}{kM_\infty^2}\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu \frac{\partial V_x}{\partial y}\right), \tag{11}$$

$$\frac{\partial(\rho V_x)}{\partial x} + \frac{\partial(\rho V_y)}{\partial y} = 0, \tag{12}$$

$$\rho V_x \frac{\partial h}{\partial x} + \rho V_y \frac{\partial h}{\partial y} = \frac{k-1}{k} V_x \frac{\partial p}{\partial x} + (k-1)M_\infty^2 \mu \left(\frac{\partial V_x}{\partial y}\right)^2 + \frac{1}{\sigma}\frac{\partial}{\partial y}\left(\mu \frac{\partial h}{\partial y}\right), \tag{13}$$

$$P = \rho h\,, \tag{14}$$

$$\mu = f(h)\,. \tag{15}$$

In system (11-15) p, h, μ are pressure, enthalpy and viscosity, respectively; k, σ, M_∞^2 are constants.

As can be seen from (11)-(15), this system of equations is not closed: six unknown functions are matched by only five equations. Assuming the mistake of Blasius, who considered the flow around the plate to be non-gradient, Dorodnitsyn et al. [14] also made an error by rejecting the pressure gradient in the momentum conservation equation in the longitudinal direction. How can a fluid flow if the plane is immobile and the pressure gradient is zero? No way for it flow. Using the experience of studying the boundary layer in an incompressible flow, as well as its development region in the boundary layer, we will perform a similar procedure for a slightly compressible flow.

It is possible to use the momentum conservation equation for the fluid flow functional over cross-section

$$J = \int_S V_x ds \to ext. \tag{16}$$

The operand has the form

$$F = V_x = V_x\left(V_y, \rho, \frac{dp}{dx}, \frac{\partial V_x}{\partial x}, \frac{\partial V_x}{\partial y}, \frac{\partial^2 V_x}{\partial y^2}, \mu, \frac{\partial \mu}{\partial y}\right). \tag{17}$$

Euler's equation, which corresponds to the necessary condition for the extreme of the functional (16) under condition (17) (this is one of the conditions: there are many more conditions, but one is enough for us), has the form:

$$\frac{\partial}{\partial x}\left(F_{u_x}\right) = \frac{\partial^2}{\partial y^2}\left(F_{u_{yy}}\right) - \frac{\partial}{\partial y}\left(F_{u_y}\right). \tag{18}$$

Following the works [1, 12], we present

$$F = V_x' = X(x) \cdot Y(y). \tag{19}$$

Substituting (19) into (18) turns (18) into equation

$$Y(y)\frac{d}{dx}\left(\frac{dX}{dx}\cdot\left(\frac{d^2X}{dx^2}\right)^{-1}\right) = X(x)\frac{d}{dy}\left(\frac{dY}{dy}\cdot\left(\frac{d^3Y}{dy^3}\right)^{-1} - \frac{dY}{dy}\cdot\left(\frac{d^2Y}{dy^2}\right)^{-1}\right). \tag{20}$$

After dividing equation (21) by the right-hand side of equation (20), we obtain:

$$\frac{1}{X(x)}\frac{d}{dx}\left(\frac{dX}{dx}\cdot\left(\frac{d^2X}{dx^2}\right)^{-1}\right) = \frac{1}{Y(y)}\frac{d}{dy}\left(\frac{dY}{dy}\cdot\left(\frac{d^3Y}{dy^3}\right)^{-1} - \frac{dY}{dy}\cdot\left(\frac{d^2Y}{dy^2}\right)^{-1}\right). \tag{21}$$

Equation (21) is a differential equation with separable variables. From the asymptotic condition, which consists in the tendency of a slightly compressible flow to an incompressible one, we assume, based on the work [1], that

$$\frac{1}{X(x)}\frac{d}{dx}\left(\frac{dX}{dx}\cdot\left(\frac{d^2X}{dx^2}\right)^{-1}\right) = 0, \quad \frac{1}{Y(y)}\frac{d}{dy}\left(\frac{dY}{dy}\cdot\left(\frac{d^3Y}{dy^3}\right)^{-1} - \frac{dY}{dy}\cdot\left(\frac{d^2Y}{dy^2}\right)^{-1}\right) = 0. \tag{22}$$

The solution of the first equation (22) is already known [1], the second solution has yet to be found. We will write it in a form convenient for solving

$$\frac{dY}{dy}\cdot\left(\frac{d^3Y}{dy^3}\right)^{-1} - \frac{dY}{dy}\cdot\left(\frac{d^2Y}{dy^2}\right)^{-1} = \mathrm{Const}\,. \tag{23}$$

An arbitrary integration constant can be chosen differently, but, as the research has shown, $\mathrm{Const} = 0$ entirely corresponds to the physics of the problem. Thus, for $\mathrm{Const} = 0$ we obtain an asymptotics in the form

$$\mathrm{Y(y) = Y(0) + D(Y)(0) \cdot y},$$

with

$$\mathrm{Y(0) = Y(y=0),\ D(Y) = \frac{dY}{dy}_{|y=0}}.$$

Since the value of the dimensionless velocity on the surface is equal to one, and the dimensionless first derivative is equal to -1, we obtain:

$$\mathrm{Y(y) = 1 - y}\,. \tag{24}$$

Therefore, for the sub-region where $y' \le 0.1$, with an accuracy of 1%, it turns out that the vertical velocity distribution near the plane is described by (24). This completely coincides with the solution for the case of incompressible fluid flow [2], since

$$\exp(-y) = 1 - y + ...$$

Thus, the previously obtained solution for the case of incompressible flow can be used as asymptotics for compressible flow - in the immediate vicinity (that is, at $y' \le 0.1$) to a solid wall.

Distribution of surface pressure in the slightly compressible flow development region in the boundary layer. The distributions of the velocity components in the flow development region have the following form [6]:

$$V_x(x,y) = \left(1 - \exp\left(\alpha x\right)\right)\exp\left(-\alpha y\right),\ V_y = \exp(\alpha x)\left(1 - \exp(-\alpha y)\right). \qquad (25)$$

According to formulas (25), neglecting the small normal component of the velocity, as well as taking into account the scale of the longitudinal velocity, we obtain for the pressure field:

$$\frac{p(x,y)}{p_0} \approx 1 - \rho V_x^2(x,y)/\left(2p_0\right) = 1 - \rho V_0^2\left(1 - \exp\left(\alpha x\right)\right)\exp\left(-2\alpha y\right)/\left(2p_0\right) \qquad (26)$$

The distribution of the pressure field is presented in Fig. 1. As one can see, the pressure field in the region of the flow development is heterogeneous. And at velocity of the order of 100 m/s, pressure deviations can be of the same order as atmospheric pressure, which affects both the lifting force and the distribution of stresses on the surface of the wing. The heterogeneity of stresses leads to the appearance of additional deformations, which, under the condition of frequent changes in flight speed, can lead to fatigue stresses, accelerating the wear of the aircraft.

Conclusions. As it turned out during the theoretical analysis, the non-slipping condition on the surface of a body moving in a immobile fluid is also not physical in the a slightly compressible fluid flow development region in the boundary layer. So, the idea appeared to expand the new concept of partial slip also at speeds at which air is already a slightly compressible fluid (80-200 m/s). Based on the use of the analytical method, which consists in the application of calculus of variations, it was found that for a slightly compressible flow, the region of its development can also be described, in close proximity to the surface of a solid body, by previously obtained functional dependencies for an incompressible flow. This affords to expand the concept of partial slip in the flow development region to the range of velocities exceeding 100 m/s. And most importantly, for this flow, the spatial distribution of pressure in the flow development region in the boundary layer should be taken into account.

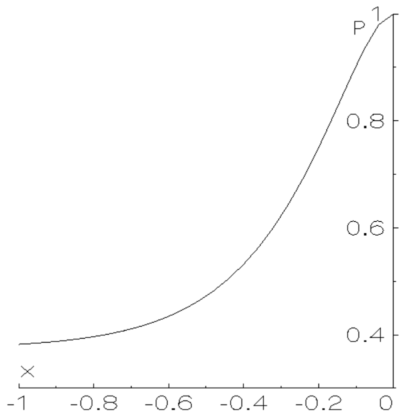

Fig.1. Dimensionless pressure distribution according to formula (26), for velocity V_0 =100 m/s.

As a future study, it is reasonable to obtain high order asymptotic for a slightly compressible flow development region in a boundary layer.

References

1. Lukianov, P.V. Flow development region in the boundary layer: two-component molecular viscosity and partial slip / P.V. Lukianov, L. Song // Aviatsiyno-kosmichna tekhnika i tekhnolohiya, 2023. 6(192). p. 38-47. doi:10.32620/aktt.2023.6.05

2. Lukianov, P.V. Optimal character and different nature of flows in laminar boundary layers of incompressible fluid flow / P.V. Lukianov, L. Song // Problems of friction and wear, 2022, 4(97), p. 52-60. doi: 10.18372/0370-2197.4(97)16959.

3. Bretheim J.U. Standard logarithmic mean velocity distribution in a band-limited restricted model of turbulent flow in a half-channel / J.U. Bretheim, C. Meneveau, D.F. Gayme // Phys. Fluids, 2015, 27, 011702.

4. Blasius H. Grenzschichten in Flussigkeiten mit kleiner Reibung / H. Blasius // Z. Math. Phys., 1908, v. 56, p. 1-37.

5. Prandtl L. Uber Flussigkeitsbewegung bei sehr kleiner Reibung / L. Prandtl // Verhandlung d. III Untern. Math. Kongr. Heidelberg, 1904, p. 484-491.

6. Karman von Th. Laminare and Turbulent Reibung / Th. von Karman // Zeitschrift für Angewandte Mathematik und Mechanik, 1921, Vol. 1, p. *233-252.*

7. Pohlhausen K. Zur Waherungsweisen Integration der Laminaren Reibungsschicht / K. Pohlhausen // Zeitschrift für Angewandte Mathematik und Mechanik , 1921, Vol. 1, p . 252--

8. Sohrab S.H. A Modified Theory of Turbulent Flow over a Flat Plate / S .H. Sohrab // Proc. Of the 5th IASME /WSEAS Conference of Fluid Mechanics and Aerodynamics, Athens, Greece, 2007, August 25-27. p. 72-80.

9. Abdul-Ghafour Q.A. A general velocity profile foe a laminar boundary layer over flat plate with zero incidence / Q.A. Abdul-Ghafour // Journal of Engineering, 2011, Vol. 17, 11 p.

10. Stokes, G.G. On the Effect of the Internal Friction of Fluids on MotioДородницын A.A. n of Pendulums / G.G. Stokes // Trans. Cambridge Philos. soc., 1851, v. 9, p. 1-106.

11. Lord Rayleigh. On the motion of solid bodies through viscous liquids / Lord Rayleigh // Phil. Mag. 1911, vol. 21, no. 6, p. 697- 711.

12. Lukianov, P.V. Unsteady incompressible laminar boundary layer: time and space variable molecular viscosity / P.V. Lukianov, L. Song // Aviatsiyno-kosmichna tekhnika i tekhnolohiya, 2023. 3(187). p. 50--60. doi: 10.32620/aktt.2023.3.06

13. Loytsyanskiy, L.G. Mekhanika zhidkosti i gaza [Mechanics of liquid and gas] 6-ye perer. I dop./ L.G. Loytsyanskiy // Moscow: Nauka, Gl. red. fiz.-mat. Lit. Publ., 1987. -- 840 p.

14. Dorodnitsyn, A.A. Pogranichnyi sloiy v szhymaemom gaze [Boundary layer in compressible gas]/ A.A. Dorodnitsyn // Prikl. mat. i mekh. [Applied math. and mech.], 1942, vo. 6. no. 6 , p. 449—486.

The article was submitted to the editorial office 14.08.2024

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ВПЛИВ ЧАСТКОВОГО ПРОКОВЗУВАННЯ НА РОЗПОДІЛ ПОВЕРХНЕВОГО ТИСКУ В ОБЛАСТІ РОЗВИТКУ ЗЛЕГКА СТИСЛИВОЇ ТЕЧІЇ В ПРИМЕЖОВОМУ ШАРІ

Дослідження ламінарної нестисливої течії рідини у примежовому шарі виявило, ще раніше, що умова повного прилипання частинок рідини до поверхні рухомого тіла (півплощини) не виконується в області розвитку (формування) течії. Припущення сталості швидкості рідини на поверхні рухомого тіла, отже повне прилипання, веде, в області розвитку течії, до повної відсутності нормальної складової поля швидкості. А це суперечить самому поняттю області розвитку течії, де повинні бути дві складові швидкості – повздовжня (основна) та нормальна (другорядна). У попередніх роботах авторів були отримані аналітичні розв'язки щодо поля швидкості у області розвитку нестисливої течії рідини у примежовому шарі. Оскільки використання моделі нестисливої течії рідини обмежується числом Маха $\mathrm{Ma} \le 0.2$, то для подальшого розширення діапазону швидкостей було розглянуто задачу про область розвитку слабко стисливої течії рідини у примежовому шарі. Аналітично доведено, що всі міркування щодо неможливості повного прилипання в області розвитку течії можна застосовувати і для слабко стисливої рідини. Слабка стисливість при цьому означає дозвуковий характер течії і нехтування температурними ефектами внаслідок тертя. На підставі критичного аналізу існуючих підходів, які розглядають обтікання рідиною нерухомої пластини у рамках без градієнтної течії (що просто неможливо через відсутність механізму створення руху рідини), показано, що система рівнянь є фактично незамкненою. Для області розвитку течії, де повздовжній градієнт тиску не є сталою величиною, не вистачає одного рівняння. Це рівняння, як і раніше у попередніх роботах, отримується із необхідної умови екстремуму функціоналу втрати рідини. І хоча повний розв'язок для повздовжньої компоненти швидкості містить чотири константи інтегрування, для отримання асимптотики поблизу твердої поверхні цілком досить знати лише дві величини – швидкість та її першу похідну. Ці величини, як виявляється із асимптотичного розв'язку, збігаються із випадком нестисливої течії, що дозволяє розширити область застосування отриманих раніше результатів для більш широкої області, наприклад $\mathrm{Ma} \le (0.5 - 0.6)$. А такі значення вже відповідають швидкостям сучасних цивільних літаків. Наводиться безрозмірний розподіл тиску в області розвитку слабко стисливої течії і показується його суттєва неоднорідність, що, в свою чергу, вказує на важливість отриманих результатів.

Ключові слова: слабко стислива течія, область розвитку течії, примежовий шар, розподіл тиску, втомні напруження та деформація поверхні, рівняння Нав'є-Стокса

Список літератури

1. Lukianov, P.V. Flow development region in the boundary layer: two-component molecular viscosity and partial slip / P.V. Lukianov, L. Song // Aviatsiyno-kosmichna tekhnika i tekhnolohiya, 2023. 6(192). p. 38-47. doi:10.32620/aktt.2023.6.05

2. Lukianov, P.V. Optimal character and different nature of flows in laminar boundary layers of incompressible fluid flow / P.V. Lukianov, L. Song // Problems of friction and wear, 2022, 4(97), p. 52-60. doi: 10.18372/0370-2197.4(97)16959.

3. Bretheim J.U. Standard logarithmic mean velocity distribution in a band-limited restricted model of turbulent flow in a half-channel / J.U. Bretheim, C. Meneveau, D.F. Gayme // Phys. Fluids, 2015, 27, 011702.

4. Blasius H. Grenzschichten in Flussigkeiten mit kleiner Reibung / H. Blasius // Z. Math. Phys., 1908, v. 56, p. 1-37.

5. Prandtl L. Uber Flussigkeitsbewegung bei sehr kleiner Reibung / L. Prandtl // Verhandlung d. III Untern. Math. Kongr. Heidelberg, 1904, p. 484-491.

6. Karman von Th. Laminare and Turbulent Reibung / Th. von Karman // Zeitschrift für Angewandte Mathematik und Mechanik, 1921, Vol. 1, p. *233-252.*

7. Pohlhausen K. Zur Waherungsweisen Integration der Laminaren Reibungsschicht / K. Pohlhausen // Zeitschrift für Angewandte Mathematik und Mechanik , 1921, Vol. 1, p . 252--

8. Sohrab S.H. A Modified Theory of Turbulent Flow over a Flat Plate / S .H. Sohrab // Proc. Of the 5th IASME /WSEAS Conference of Fluid Mechanics and Aerodynamics, Athens, Greece, 2007, August 25-27. p. 72-80.

9. Abdul-Ghafour Q.A. A general velocity profile foe a laminar boundary layer over flat plate with zero incidence / Q.A. Abdul-Ghafour // Journal of Engineering, 2011, Vol. 17, 11 p.

10. Stokes, G.G. On the Effect of the Internal Friction of Fluids on MotioДородницын A.A. n of Pendulums / G.G. Stokes // Trans. Cambridge Philos. soc., 1851, v. 9, p. 1-106.

11. Lord Rayleigh. On the motion of solid bodies through viscous liquids / Lord Rayleigh // Phil. Mag. 1911, vol. 21, no. 6, p. 697- 711.

12. Lukianov, P.V. Unsteady incompressible laminar boundary layer: time and space variable molecular viscosity / P.V. Lukianov, L. Song // Aviatsiyno-kosmichna tekhnika i tekhnolohiya, 2023. 3(187). p. 50--60. doi: 10.32620/aktt.2023.3.06

13. Loytsyanskiy, L.G. Mekhanika zhidkosti i gaza [Mechanics of liquid and gas] 6-ye perer. I dop./ L.G. Loytsyanskiy // Moscow: Nauka, Gl. red. fiz.-mat. Lit. Publ., 1987. -- 840 p.

14. Dorodnitsyn, A.A. Pogranichnyi sloiy v szhymaemom gaze [Boundary layer in compressible gas]/ A.A. Dorodnitsyn // Prikl. mat. i mekh. [Applied math. and mech.], 1942, vo. 6. no. 6 , p. 449—486.

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