AUTOMATIC CONTROL SYSTEMS

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ATTITUDE DETERMINATION USING DISTANCES MEASUREMENTS

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Abstract*—The attitude determination based on measuring the distances between points of the body and reference points (beacons) is analyzed. The conditions for choosing these points are determined. In the analysis, the matrix method and the least squares method were used. As part of the problem, a positioning problem of independent significance has been solved. The influence of distance measurement errors on the accuracy of determining the orientation is investigated. The proposed form of the loss function allows us to simplify the expression for the matrix of directional cosines and, thereby, perform an analytical analysis of errors. It is shown that for small errors, their influence on the accuracy of determining orientation angles can be characterized by an additional rotation of the coordinate system associated with the body. Theoretical and numerical analysis showed that the accuracy of determining the orientation can be significantly increased by increasing the number of measurements.*

Index Term—distance; attitude; determination.

I. INTRODUCTION

Measurement of distances between fixed points and points of a moving body is the basis of algorithms for determining the orientation using Global Positioning System (GPS) [1], [2]. But GPS can't be always used due to technical restrictions on its use. The development of modern information technologies has led to widespread use for determining the distances of ultrasound and radio methods. That is, they are used to solve the problem of positioning a moving body [3]. Consider the measurement of distances between fixed points and moving points for solving the problem of determining the orientation of a moving body.

II. PROBLEM STATEMENT

Consider the problem of determining the orientation of the body based on information about the position of fixed points (beacons) $A_i (i = 1, ..., n)$ and distances between these points and the points of the moving body (nodes) $M_j (j = 1, ..., m)$ (Fig. 1). The position of fixed points A_i with known coordinates is given by the vectors $\vec{r}_i$. The position of moving points M_j with unknown coordinates is given by the vectors $\vec{\rho}_j$. The position of the point A_i relative to the point M_j is given by the vector $\vec{d}_{ji}$.

From the triangle OM_jA_i find

$$d_{ji}^2 = r_i^2 + \rho_j^2 - 2\boldsymbol{r}_i^{\mathrm{T}} \boldsymbol{\rho}_j. \qquad (1)$$

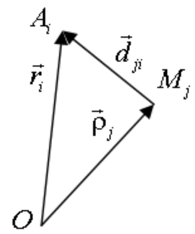

Fig. 1. The points and the vectors

To exclude from consideration a nonlinear unknown component ρ_j^2, write the difference

$$d_{ji}^2 - d_{j1}^2 = r_i^2 - r_1^2 - 2(\boldsymbol{r}_i - \boldsymbol{r}_1)^{\mathrm{T}} \boldsymbol{\rho}_j.$$

That is,

$$\boldsymbol{k}_i \boldsymbol{\rho}_{ji} = h_{ji}, \qquad (2)$$

where

$$\boldsymbol{k}_i = 2(\boldsymbol{r}_i - \boldsymbol{r}_1)^{\mathrm{T}}, \quad h_{ji} = d_{j1}^2 - d_{ji}^2 + r_i^2 - r_1^2.$$

From one equation (2) it is impossible to find a vector $\vec{\rho}_j$ so add the same relation for other points A_i and consider the system of equations

$$K\boldsymbol{\rho}_j = h_j, \qquad (3)$$

where

$$\mathbf{K} = 2\begin{bmatrix} (\boldsymbol{r}_2 - \boldsymbol{r}_1)^{\mathrm{T}} \\ (\boldsymbol{r}_3 - \boldsymbol{r}_1)^{\mathrm{T}} \\ \vdots \\ (\boldsymbol{r}_n - \boldsymbol{r}_1)^{\mathrm{T}} \end{bmatrix}, \quad \boldsymbol{h}_j = \begin{bmatrix} d_{j1}^2 - d_{j2}^2 + r_2^2 - r_1^2 \\ d_{j1}^2 - d_{j3}^2 + r_3^2 - r_1^2 \\ \vdots \\ d_{j1}^2 - d_{jn}^2 + r_n^2 - r_1^2 \end{bmatrix}.$$

From here find a vector ρ_j that characterizes the translational movement of the body. If, $n = 4$ that is, the matrix $\mathbf{K}$ is square, then

$$\rho_j = \mathbf{K}^{-1}\boldsymbol{h}_j. \tag{4}$$

The need to find an inverse matrix $\mathbf{K}^{-1}$ imposes a restriction on the choice of vectors $\vec{r}_i$. The determinant of the matrix $\mathbf{K}$ will be zero in the case when at least one of the vectors $\boldsymbol{r}_2 - \boldsymbol{r}_1$, $\boldsymbol{r}_3 - \boldsymbol{r}_1$, $\boldsymbol{r}_4 - \boldsymbol{r}_1$ will be proportional to another, that is, these vectors are parallel. This case should be excluded. Suppose, $\frac{\boldsymbol{r}_3 - \boldsymbol{r}_1}{\boldsymbol{r}_2 - \boldsymbol{r}_1} = \lambda$ that is $\boldsymbol{r}_3 = \boldsymbol{r}_1 + \lambda(\boldsymbol{r}_2 - \boldsymbol{r}_1)$. In this case, the vectors $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\boldsymbol{r}_3$ are in the same plane. Thus, in order that the determinants of the matrix $\mathbf{K}$ do not equal zero, it is sufficient to choose the stationary vectors in such a way that all vectors are not in the same plane.

If $n > 4$, i.e. the matrix $\mathbf{K}$ is not square, the vector ρ_j can be found by the formula

$$\rho_j = Dh_j. \tag{5}$$

where $D = (K^{\mathrm{T}}K)^{-1}K^{\mathrm{T}}$, $D = \left(K^{\mathrm{T}}K\right)^{-1}K^{\mathrm{T}}$.

Note that this expression is optimal in terms of the minimum error of estimating a vector ρ_j.

To determine the orientation of the body write the expressions for the vectors associated with the body

$$\boldsymbol{u}_s = \rho_s - \rho_1 = D(\boldsymbol{h}_s - \boldsymbol{h}_1) = D\boldsymbol{h}_{s*} \quad s = 2 \pm m, \tag{6}$$

where

$$\boldsymbol{h}_{s*} = \begin{bmatrix} d_{s1}^2 - d_{s2}^2 - (d_{11}^2 - d_{12}^2) \\ d_{s1}^2 - d_{s3}^2 - (d_{11}^2 - d_{13}^2) \\ \vdots \\ d_{s1}^2 - d_{sn}^2 - (d_{11}^2 - d_{1n}^2) \end{bmatrix}.$$

Combine the equation into a system

$$KU = H, \tag{7}$$

where $U = [\boldsymbol{u}_2 \ \boldsymbol{u}_3 ... \boldsymbol{u}_m]$; $H = [\boldsymbol{h}_{1*} \ \boldsymbol{h}_{2*} \ ... \ \boldsymbol{h}_{m-1*}]$.

Consider the use of least square method to determine the matrix of directional cosines R, which establishes the relationship between the projections of any vector before ($\boldsymbol{u}_{s0}$) and after ($\boldsymbol{u}_s$)the turn of the body ($\boldsymbol{u}_s = R\boldsymbol{u}_{s0}$).

The task of estimating the orientation using the method of least squares [4] is to determine the orthogonal matrix $\mathbf{R}$ with the determinant +1, which would minimize the loss function which assume in the form

$$\begin{aligned} g_1(R) &= \frac{1}{2}\sum_{v=1}^{L} \mathrm{tr}\Big\{\left[GK^{\mathrm{T}}(KRU_0 - H_v)E\right]^{\mathrm{T}} \\ &\qquad \cdot\left[GK^{\mathrm{T}}(KRU_0 - H_v)E\right]\Big\} \\ &= \frac{1}{2}\sum_{v=1}^{L} \mathrm{tr}\Big(E^{\mathrm{T}}U_0^{\mathrm{T}}U_0E - 2E^{\mathrm{T}}H_v^{\mathrm{T}}KG^{\mathrm{T}}RU_0E \\ &\qquad + E^{\mathrm{T}}H_v^{\mathrm{T}}KG^{\mathrm{T}}GK^{\mathrm{T}}H_vE\Big), \end{aligned} \tag{8}$$

where L are number of measurements; $G = \left(K^{\mathrm{T}}K\right)^{-1}$; $\mathbf{E}$ is the matrix, the expediency of which will be explained later.

Use the restrictions $R^{\mathrm{T}}R = I$ and take the loss function in the form

$$g_2 = g_1 + \mathrm{tr}\left(\frac{1}{2}\mathbf{\Lambda}\left(R^{\mathrm{T}}R - I\right)\right), \tag{9}$$

where $\mathbf{\Lambda}$ is the matrix (Lagrange multiplier).

Equating the derivative g_2 over R to zero, obtain:

$$Q = \frac{1}{2}\left(R\mathbf{\Lambda}^{\mathrm{T}} + R\mathbf{\Lambda}\right), \tag{10}$$

where $Q = GK^{\mathrm{T}}H_LEE^{\mathrm{T}}U_0^{\mathrm{T}}$, $H_L = \sum_{v=1}^{L} H_v$.

Let's assume that the matrix $\mathbf{\Lambda}$ is symmetric. Then formula (10) takes the form $Q = R\mathbf{\Lambda}$ or $R = Q\mathbf{\Lambda}^{-1}$. Substituting this expression into a relationship $R^{\mathrm{T}}R = I$, find $\mathbf{\Lambda} = \sqrt{Q^{\mathrm{T}}Q}$, that is, the calculated matrix will be equal

$$R_r = Q\left(\sqrt{Q^{\mathrm{T}}Q}\right)^{-1}. \tag{11}$$

To find the matrix $\mathbf{E}$, consider the expression $Q^{\mathrm{T}}Q$ in the absence of measurement errors for one measurement. In this case

$$V = Q^{\mathrm{T}}Q = \mathbf{U}_0\mathbf{E}\mathbf{E}^{\mathrm{T}}\mathbf{U}_0^{\mathrm{T}}\mathbf{U}_0\mathbf{E}\mathbf{E}^{\mathrm{T}}\mathbf{U}_0^{\mathrm{T}}. \tag{12}$$

Choose the matrix $\mathbf{E}$ from the condition $V = I$. Introduce the notation $E_1 = EE^{\mathrm{T}}$, $X = U_0^{\mathrm{T}} U_0$ and multiply the expression (12) first to the left onto the matrix $\mathbf{U}_0^{\mathrm{T}}$, and then to the right to the matrix $\mathbf{U}_0$. Then $XE_1 = I$ and $EE^{\mathrm{T}} = \left(U_0^{\mathrm{T}} U_0\right)^{-1}$. Accept $E = E^{\mathrm{T}}$. Then $E = \sqrt{\left(U_0^{\mathrm{T}} U_0\right)^{-1}}$. Finally

$$Q = GK^{\mathrm{T}} H_L \left(U_0^{\mathrm{T}} U_0\right)^{-1} U_0^{\mathrm{T}}. \tag{13}$$

and

$$\begin{aligned} \psi &= \mathrm{arctg}\left(R_r(1,2) / R_r(1,1)\right), \\ \theta &= -\arcsin\left(R_r(1,3)\right), \\ \varphi &= \mathrm{arctg}\left(R_r(2,3) / R_r(3,3)\right). \end{aligned} \tag{14}$$

The introduction of the matrix $\mathbf{E}$ allows obtain analytical expressions for errors in the determination of angles.

Considering the measurement errors of small ones, write the matrix $\mathbf{H}_v$ in the form $\mathbf{H}_\nu \approx \mathbf{H}_* + \boldsymbol{\Delta}_{\mathbf{H}\nu}$, where $\mathbf{H}_* = KRU_0$ is the matrix $\mathbf{H}$ in the absence of errors of measurements; $\boldsymbol{\Delta}_{\mathbf{H}\nu}$ is the matrix of errors. Then

$$\begin{aligned} Q &= GK^{\mathrm{T}} \left(\sum_{\nu=1}^{L} \left(\mathbf{H}_* + \boldsymbol{\Delta}_{\mathbf{H}\nu}\right) \right) \mathbf{E}_1 \mathbf{U}_0^{\mathrm{T}} \\ &= LR + GK^{\mathrm{T}} \Delta_{HL} \mathbf{E}_1 \mathbf{U}_0^{\mathrm{T}}, \end{aligned} \tag{15}$$

where $\Delta_{HL} = \sum_{\nu=1}^{L} \Delta_{\mathbf{H}\nu}$.

Then

$$\begin{aligned} Q^{\mathrm{T}} Q &= \left(LR^{\mathrm{T}} + U_0 E_1 \Delta_{HL}^{\mathrm{T}} KG^{\mathrm{T}}\right)\left(LR + GK^{\mathrm{T}} \Delta_{HL} E_1 U_0^{\mathrm{T}}\right) \\ &\approx L^2 I + L\left(V + V^{\mathrm{T}}\right), \end{aligned}$$

where

$$V = R^{\mathrm{T}} GK^{\mathrm{T}} \Delta_{HL} E_1 U_0^{\mathrm{T}}.$$

For small errors you can write down

$$\left(\sqrt{Q^{\mathrm{T}} Q}\right)^{-1} \approx L^{-1} \left[I - \frac{1}{2L}\left(V + V^{\mathrm{T}}\right) \right].$$

Then the matrix $\mathbf{R}_r$ (11) will be equal to

$$\mathbf{R}_r = LR\left(I + \frac{1}{L} V \right) L^{-1} \left(I - \frac{1}{2L}\left(V + V^{\mathrm{T}}\right) \right) \approx BR, \tag{16}$$

where $B = I + \frac{1}{2}\left(\tilde{Z} - \tilde{Z}^{\mathrm{T}}\right)$, $\tilde{Z} = GK^{\mathrm{T}} \tilde{\Delta}_H E_1 U_0^{\mathrm{T}} R^{\mathrm{T}}$; $\tilde{\Delta}_H = \frac{1}{L} \sum_{\nu=1}^{L} \boldsymbol{\Delta}_{H\nu}$ is the averaged value of the matrix $\boldsymbol{\Delta}_{H\nu}$.

The matrix $\mathbf{B}$, which is the sum of the unit and small skew-symmetric matrix, can be considered as an additional rotation matrix (matrix of errors of determination of angles) due to measurement errors. That is, you can write

$$B = I + \begin{vmatrix} 0 & \varepsilon_z & -\varepsilon_y \\ -\varepsilon_z & 0 & \varepsilon_x \\ \varepsilon_y & -\varepsilon_x & 0 \end{vmatrix}, \tag{17}$$

where ε_x, ε_y, ε_z is the orientation error, which is calculated relative to the axes of the connected coordinate system.

The relationship between the errors calculated in the associated coordinate system ε_x, ε_y, ε_z and the errors $\Delta\psi$, Δ_θ, Δ_φ calculated in the reference coordinate system (the difference between the calculated and set values of the angles), the following [2]:

$$\begin{aligned} \Delta\psi &= \frac{1}{\cos\theta}\left(\varepsilon_z \cos\varphi + \varepsilon_y \sin\varphi\right), \\ \Delta_\theta &= \varepsilon_y \cos\varphi - \varepsilon_z \sin\varphi, \\ \Delta_\varphi &= \varepsilon_x + \mathrm{tg}\,\theta(\varepsilon_z \cos\varphi + \varepsilon_y \sin\varphi). \end{aligned} \tag{18}$$

Perform a numerical error analysis.

Assign (given in meters) the position of four fixed points $\boldsymbol{r}_1 = [10\ -8\ 9]^{\mathrm{T}}$, $\boldsymbol{r}_2 = [8\ -8\ 10]^{\mathrm{T}}$, $\boldsymbol{r}_3 = [9\ 8\ 10]^{\mathrm{T}}$, $\boldsymbol{r}_4 = [10\ 9\ -8]^{\mathrm{T}}$ and the point of the body $\boldsymbol{\rho}_1 = [0.4\ 0.6\ -0.3]^{\mathrm{T}}$.

To form the displacement of three more points, set their position in the body relative to the first point of the body (that is, the beginning of the moving coordinate system is taken at the first point of the moving body) $\boldsymbol{u}_{2n} = [0.2\ 0.3\ -0.4]^{\mathrm{T}}$, $\boldsymbol{u}_{3n} = [0.5\ 0.3\ 0.4]^{\mathrm{T}}$, $\boldsymbol{u}_{4n} = [0.3\ 0.5\ -0.4]^{\mathrm{T}}$ and the Euler's three corners of the body rotation $\psi = 10°$, $\theta = 20°$, $\varphi = 30°$.

Note that the choice of moving vectors $\boldsymbol{u}_2, ..., \boldsymbol{u}_m$ is subject to the same condition as the choice of fixed vectors.

The current position of moving points was calculated by the formula $\rho_j = \rho_1 + R\boldsymbol{u}_{jn}$, $j = 2, 3, 4$. For measurement errors, a normal distribution law has been adopted, namely, taken

$$d_{ij} = \left\| \vec{r}_i - \vec{\rho}_j \right\| \left(1 + 1 \cdot 10^{-4} M(i,j)\right),$$

where $d_{ij} = \left\| \vec{r}_i - \vec{\rho}_j \right\| \left(1 + 1 \cdot 10^{-4} M(i,j)\right)$ is the matrix of random numbers.

Figure 2 and Figure 3 shows the dependence of the number of averaging of the errors of the Euler angle determination as the difference between the values calculated by the formula (11) and the given values.

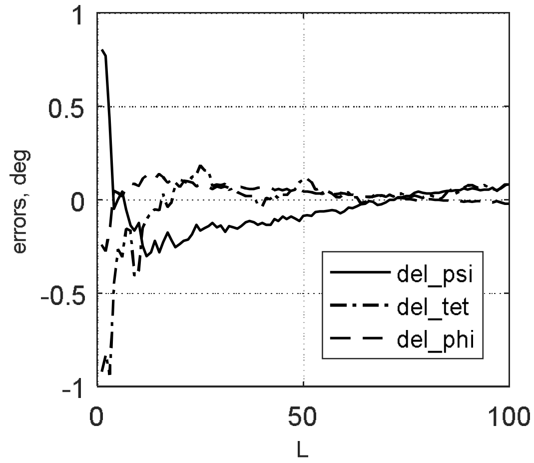

Fig. 2. Errors of angles determination

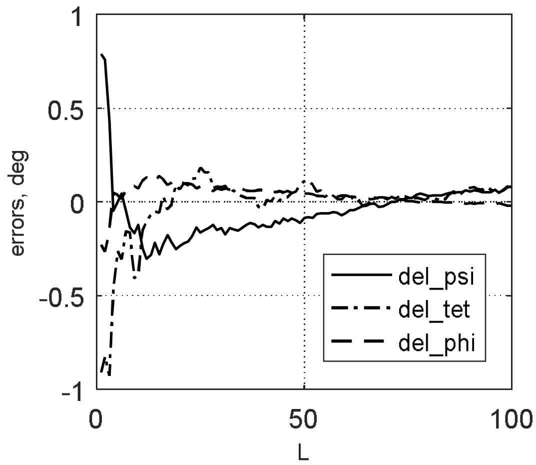

Fig. 3. Errors of angles determination

The graphs of errors, calculated by the formulas (18), practically coincide with the given graphs.

One way to improve the accuracy of determining the angle orientation when using GPS, is to increase the distance between receivers [2]. Show that a similar result holds in this system. For analysis, the vectors $\boldsymbol{u}_{2n}$, $\boldsymbol{u}_{3n}$, $\boldsymbol{u}_{4n}$, were multiplied by q times. Figure 4 shows the dependence of the parameter q (at $L=1$) relative error of the determination of angles $\lambda_\psi = \frac{\Delta_{\psi_q}}{\Delta_{\psi_{q=1}}}$, $\lambda_\theta = \frac{\Delta_{\theta_q}}{\Delta_{\theta_{q=1}}}$, $\lambda_\varphi = \frac{\Delta_{\varphi_q}}{\Delta_{\varphi_{q=1}}}$. They are identical and very close to dependence $\frac{1}{q}$. The graphs of errors, calculated by the formulas (18), practically coincide with the given graphs.

Definition of displacement $\vec{\rho}_1$ s of independent importance (this is a solution to the problem of positioning the body). Figure 5 shows the errors of determining the coordinates of moving the point M_1 as the difference between the values (5) averaged over the L measurements and the given values. Averaged values were searched by the

$$\tilde{\rho}_1 = D\tilde{h}_1, \qquad (19)$$

where $\tilde{h}_1 = \frac{1}{L}\sum_{\nu=1}^{L} h_{1\nu}$.

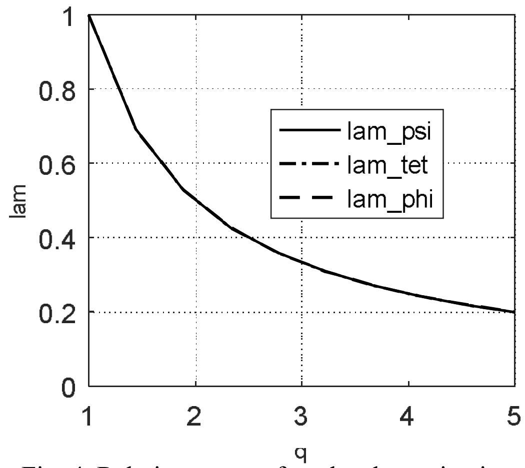

Fig. 4. Relative errors of angles determination

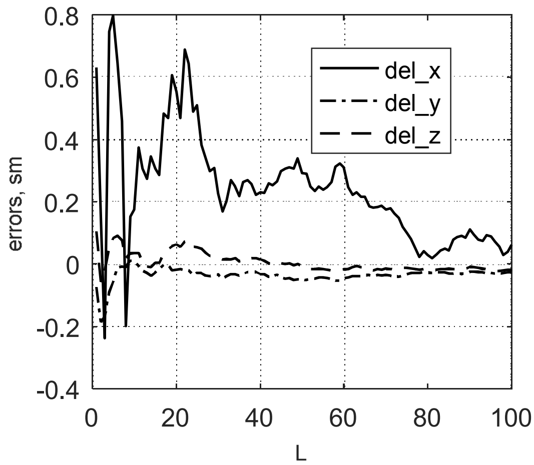

Fig. 5. Errors of point M_1 coordinates determination

III. Conclusion

An algorithm for determining the orientation of a solid on the basis of measuring the distances between moving points of the body and fixed points

is proposed. The efficiency of using the matrix form of the least squares method for determining the orientation is shown. The selected loss function allowed perform an analytical study of orientation error errors based on the use of an additional rotation matrix.

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Л. М. Рижков. Визначення орієнтації на основі вимірювання відстаней

Проаналізовано визначення орієнтації на основі вимірювання відстаней між точками тіла та опорними точками (маяками). Визначено умови вибору цих точок. При аналізі використано матричний метод та метод найменших квадратів. Як частину задачі розв'язано задачу позиціювання, яка має самостійне значення. Досліджено вплив похибок вимірювання відстаней на точність визначення орієнтації. Запропонована форма функції втрат дозволяє спростити вираз для матриці напрямних косинусів і, тим самим, виконати аналітичний аналіз похибок. Показано, що для малих похибок їх вплив на точність визначення кутів орієнтації можна характеризувати додатковим поворотом зв'язаної з тілом системи координат. Теоретичний та чисельний аналіз показав, що точність визначення орієнтації можна суттєво підвищити за рахунок збільшення кількості вимірювань.
Ключові слова: відстань; положення; визначення.

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Л. М. Рыжков. Определение ориентации на основе измерения расстояний

Проанализировано определение ориентации на основе измерения расстояний между точками тела и опорными точками (маяками). Определены условия выбора этих точек. При анализе использован матричный метод и метод наименьших квадратов. Как часть задачи решена задача позиционирования, имеющая самостоятельное значение. Исследовано влияние погрешностей измерения расстояний на точность определения ориентации. Предложенная форма функции потерь позволяет упростить выражение для матрицы направляющих косинусов и, тем самым, выполнить аналитический анализ погрешностей. Показано, что для малых погрешностей их влияние на точность определения углов ориентации можно характеризовать дополнительным поворотом связанной с телом системы координат. Теоретический и численный анализ показал, то точность определения ориентации можно существенно повысить за счет увеличения количества измерений.
Ключевые слова: расстояние; положение; определение.

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