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SYNTHESIS AND ANALYSIS OF COMPLEMENTARY FILTER FOR ATTITUDE DETERMINATION

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Abstract*—The problems of synthesis and analysis of complementary filters used for determination of the moving vehicle attitude are considered. The paper deals with the complementary filter operating in an arbitrarily located reference frame determined by specific functioning of the measuring device. The transfer functions of the controllers in the correction channels of accelerometer and magnetometer are represented. It is shown that using a gyroscope and an accelerometer is accompanied with a drift relative to all axes of the reference frame. Relationships for calculating these drifts have been obtained. The simplified equations of the complementary filter relative to the orientation error were derived. The influence of the structure of the transfer function of the controller in the correction channel by the accelerometer signals on the static accuracy of the complementary filter is analyzed. It is shown that transformation of the angles to the coordinate system with a vertical axis allows eliminating the static errors caused by the magnetometer relative to the horizontal axes, at that an error relative to the vertical axis remains.*

Index Terms—Complementary filter; attitude; gyroscope; accelerometer; magnetometer.

I. INTRODUCTION

Complementary filters are effective means to improve the accuracy of determining the attitude of moving vehicles [1] – [3]. Most often, the integration of the angular rate sensor and accelerometer leads to the effect of the gyroscopic drift. In order to increase the efficiency of the complementary filter, a magnetometer [4] is additionally used. As a rule, the analysis is performed in the coordinate system (we will denote it as $OX_oY_oZ_o$), where one axis is directed along the local vertical, and the other two axes are arranged in the horizon plane.

II. PROBLEM STATEMENT

Let us consider a problem of the synthesis of the complementary filter in a reference coordinate system $OX_oY_oZ_o$ with the vertical axis OZ_o. A position of the vehicle is determined in the body-axis coordinate system $OXYZ$ by means of a quaternion q (angles ψ, θ, φ).

In addition, we will use two separate controllers: for measuring channels with accelerometer and magnetometer respectively.

The flow diagram of a complementary filter is shown in Fig. 1.

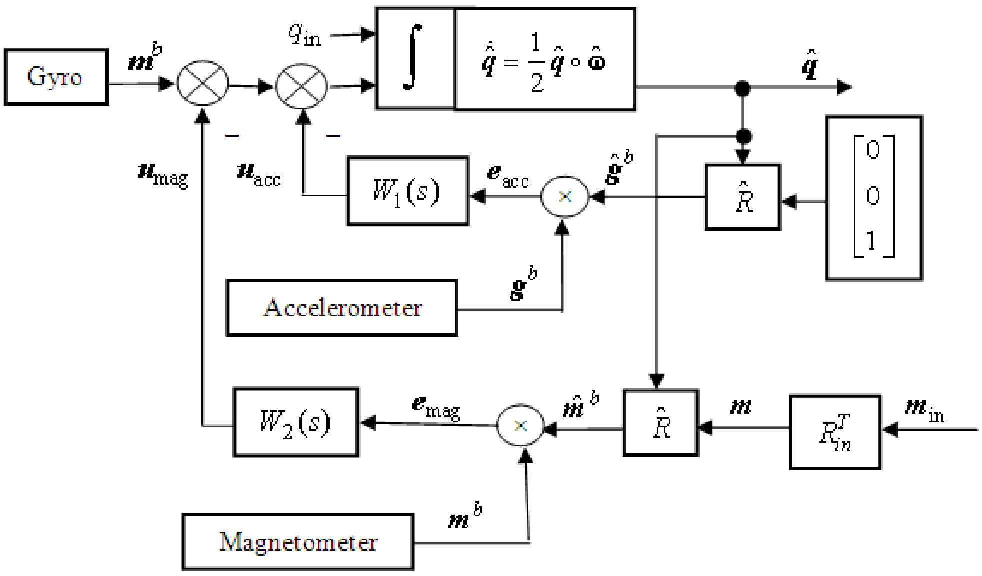

Fig. 1. Flow diagram of the complementary filter

The main goal of the research is to derive model of errors and to determine transfer functions of controllers in measuring channels.

III. PROBLEM SOLUTION

Let us write the equation of motion of a filter in the following form

$$\dot{\hat{q}} = \frac{1}{2}\hat{q} \circ \hat{\omega},$$
$$\boldsymbol{e}_{\text{acc}} = \hat{\boldsymbol{g}}^b \times \boldsymbol{g}^b,\ \boldsymbol{e}_{\text{mag}} = \hat{\boldsymbol{m}}^b \times \boldsymbol{m}^b, \quad (1)$$
$$\boldsymbol{u}_{\text{acc}} = W_1(s)\boldsymbol{e}_{\text{acc}},\ \boldsymbol{u}_{\text{mag}} = W_2(s)\boldsymbol{e}_{\text{mag}},$$

where $\boldsymbol{m}_{\text{in}}$ is the remembered at the initial instant of time normalized output signal of the magnetometer; $\hat{\omega} = \tilde{\omega} + \boldsymbol{u}$; $\boldsymbol{u} = \boldsymbol{u}_{\text{acc}} + \boldsymbol{u}_{\text{mag}}$; $\tilde{\omega} = \omega + \delta_\omega$; ω is the real value of the angular velocity; δ_ω is a drift of a gyroscope; $\hat{\boldsymbol{g}}^b = \boldsymbol{g}^b + \boldsymbol{\delta}_{\text{acc}}$; $\boldsymbol{g}^b$ is the real value of the projections of the normalized vector $\boldsymbol{g}$ on the axes of sensitivity of the accelerometer; $\boldsymbol{\delta}_{\text{acc}}$ are errors of an accelerometer; $\hat{\boldsymbol{m}}^b = \boldsymbol{m}^b + \boldsymbol{\delta}_{\text{mag}}$; $\boldsymbol{m}^b$ is the real value of the projections of the normalized vector $\boldsymbol{m}$ of the intensity of the magnetic field of the Earth on the magnetometer's axes; $\boldsymbol{\delta}_{\text{mag}}$ are errors of a magnetometer; q_{in}, $\boldsymbol{m}_{\text{in}}$, R_{in} is the initial values of the quaternion of orientation, the magnetic moment and the directional cosine matrix.

Denoting $\boldsymbol{h} = \boldsymbol{g}^b$; $\hat{\boldsymbol{h}} = \hat{\boldsymbol{g}}^b$ and analyzing the expression $\boldsymbol{e} = \hat{\boldsymbol{h}} \times \boldsymbol{h}$ we will consider the structure of the signals $\boldsymbol{e}_{\text{acc}}$ and $\boldsymbol{e}_{\text{mag}}$. in more detail.

At the initial instant of time, the angles θ_{in} and φ_{in} are determined using the accelerometer. The angle ψ_{in} is assumed to be zero. This allows us to find the corresponding matrix R_{in} and quaternion q_{in}. Quaternion q_{in} is used to determine the initial conditions for integrating Euler's kinematic equations.

At the initial moment of time, using the matrix R_{in}^{T}, the magnetic moment $\boldsymbol{m}_{\text{in}}$ measured by a magnetometer is converted into coordinate system $OX_oY_oZ_o$.

To decrease errors in the determination of the orientation angles, it is possible to align an additional rotation relatively to the coordinate system $OXYZ$ at small angles $\sigma_x, \sigma_y, \sigma_z$. This rotation will be characterized by a quaternion $\delta_q \approx 1 + \frac{1}{2}\boldsymbol{\sigma}$ and a matrix

$$\Xi = \begin{bmatrix} 1 & \sigma_z & -\sigma_y \\ -\sigma_z & 1 & \sigma_x \\ \sigma_y & -\sigma_x & 1 \end{bmatrix},$$

where $\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \sigma_y & \sigma_z \end{bmatrix}^{\text{T}}$ is the error vector.

There are the relations

$$\hat{q} \approx q \circ \delta_q;\ \hat{\boldsymbol{h}} = \Xi\boldsymbol{h} + \boldsymbol{\delta}_h;\ \boldsymbol{e} = H\hat{\boldsymbol{h}}, \quad (2)$$

where

$$H = \begin{bmatrix} 0 & h_z & -h_y \\ -h_z & 0 & h_x \\ h_y & -h_x & 0 \end{bmatrix},\quad \boldsymbol{h} = \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix}.$$

The expression for the vector $\boldsymbol{e}$ will be written in the form

$$\boldsymbol{e} = H\left(\Xi\boldsymbol{h} + \boldsymbol{\delta}_h\right) = M\boldsymbol{\sigma} + H\boldsymbol{\delta}_h, \quad (3)$$

where

$$M = -H^2 = \begin{bmatrix} h_y^2 + h_z^2 & -h_xh_y & -h_xh_z \\ -h_yh_x & h_x^2 + h_z^2 & -h_yh_z \\ -h_zh_x & -h_zh_y & h_x^2 + h_y^2 \end{bmatrix}.$$

That is, the vector $\boldsymbol{e}$ is proportional to the vector of errors $\boldsymbol{\sigma}$.

The conclusion about the choice of the control law can be generalized. Consider some dependencies. The vector product $\vec{c} = \vec{a} \times \vec{b}$ of the vectors $a = [a_x\ a_y\ a_z]^{\text{T}}$, $b = [b_x\ b_y\ b_z]^{\text{T}}$ in the matrix form is $c = Ab$, where

$$A = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}.$$

Then the vector product $\vec{d} = \vec{a} \times \vec{c} = \vec{a} \times \left(\vec{a} \times \vec{b}\right)$ in the matrix form can be written as follows

$$d = Ac = A\,Ab = A^2b.$$

The device's value of the vector $\vec{h}$ in the body-axis coordinate system is equal to $\vec{h}_b = \hat{R}\vec{h}$, where $\hat{R}$ is the calculated matrix of the directional cosines; $\vec{h}$ is the vector in the reference coordinate system.

For small errors can write down

$$\hat{R} = \Delta_R R = \left(I + \Xi\right)R.$$

Then $\hat{h}_b = (I+\Xi)Rh = Rh + \Xi Rh = h_{b*} + \Xi h_{b*}$,
where $h_{b*} = Rh$ is the ideal value of the vector $\vec{h}_b$.

In the vector form, this expression will be written as follows

$$\vec{\hat{h}}_{b*} = \vec{h}_{b*} - \vec{\sigma} \times \vec{h}_{b*}.$$

We will find a product

$$\vec{e} = \vec{\hat{h}}_{b*} \times \vec{h}_{b*} = \left(\vec{h}_{b*} - \vec{\sigma} \times \vec{h}_{b*}\right) \times \vec{h}_{b*}$$
$$= -\left(\vec{\sigma} \times \vec{h}_{b*}\right) \times \vec{h}_{b*} = -\vec{h}_{b*} \times \left(\vec{h}_{b*} \times \vec{\sigma}\right).$$

Then we can write

$$\boldsymbol{e} = -G^2 \boldsymbol{\sigma}, \tag{4}$$

where $G = \begin{bmatrix} 0 & h_{b*z} & -h_{b*y} \\ -h_{b*z} & 0 & h_{b*x} \\ h_{b*y} & -h_{b*x} & 0 \end{bmatrix}$.

From this follows that the correction signal is proportional to the error vector.

Consider a separate case when the vector $\boldsymbol{h}$ has a non-zero projection on only one axis of the coordinate system, for example, on the axis OZ_o (this holds for a vector $\vec{\mathbf{g}}$). In this case

$$\boldsymbol{e} = h_z^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{bmatrix} + H\boldsymbol{\delta}_h = h_z^2 \begin{bmatrix} \sigma_x \\ \sigma_y \\ 0 \end{bmatrix} + H\boldsymbol{\delta}_h. \tag{5}$$

In this case, there is no component of the correction signal, the proportions of the angle σ_z, that is, there is no correction in the horizontal plane.

We consider the given angular velocity of drift $\omega_x^\partial, \omega_y^\partial, \omega_z^\partial$ around the axes of the body coordinate system $OXYZ$. There will be a drift relative to the axis OZ_o and there will be no drift relative to the axes OX_o and OY_o. That is, the resulting vector of drift $\boldsymbol{\delta}_{\omega o}$ in the coordinate system $OX_oY_oZ_o$ will have the form

$$\boldsymbol{\delta}_\omega^o = KR^{\mathrm{T}}\boldsymbol{\delta}_\omega^b, \tag{6}$$

where $K = [0\ 0\ 0;\ 0\ 0\ 0;\ 0\ 0\ 1]$.

This vector in some coordinate system $OX_rY_rZ_r$, position of which is determined by the matrix $\mathbf{A}$, will have

$$\boldsymbol{\delta}_\omega^r = A\boldsymbol{\delta}_\omega^o. \tag{7}$$

The physical meaning of this expression is as follows. We must find the projection of the vector of the angular velocity of drift $\boldsymbol{\delta}_\omega^b$ on the vertical axis (vector $\boldsymbol{\delta}_\omega^o$), and then must rewrite this vector in the coordinate system $OX_rY_rZ_r$.

That is, if the complementary filter is used in the coordinate system $OX_rY_rZ_r$ in the presence of correction only from the accelerometer, there will be a drift relative to all axes of this coordinate system.

To design the control laws, we will perform a system analysis (1) relative to the variable $\boldsymbol{\sigma}$. As this variable is specified in rotational coordinate system $OXYZ$ we can write $\dot{\vec{\sigma}} + \vec{\omega} \times \vec{\sigma} = \vec{\delta}_\omega + \vec{u}$ or

$$\dot{\boldsymbol{\sigma}} + \Omega\boldsymbol{\sigma} = \boldsymbol{\delta}_\omega + \mathbf{u}, \tag{8}$$

where $\Omega = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$.

That is, in the analysis of errors in the system of equations (1), the first equation can be replaced by equation (8) for analyzing the system with respect to errors.

Combine expressions (4) and (8)

$$\left(sI + D\right)\boldsymbol{\sigma} = \boldsymbol{\delta}_\omega - W_1(s)H_{\text{acc}}\boldsymbol{\delta}_{\text{acc}} - W_2(s)H_{\text{mag}}\boldsymbol{\delta}_{\text{mag}}, \tag{9}$$

where $D = W_1(s)M_{\text{acc}} + W_2(s)M_{\text{mag}} + \Omega$.

Let's accept

$$W_1(s) = W_2(s) = k_P + \frac{k_I}{s}. \tag{10}$$

Then expression (10) can be written as follows

$$\left(s^2 I + D_1\right)\boldsymbol{\sigma} = s\boldsymbol{\delta}_\omega - \left(k_P s + k_I\right)\left(H_{\text{acc}}\boldsymbol{\delta}_{\text{acc}} + H_{\text{mag}}\boldsymbol{\delta}_{\text{mag}}\right), \tag{11}$$

where $D_1 = \left(k_P s + k_I\right)\left(M_{\text{acc}} + M_{\text{mag}}\right) + s\Omega$.

We see that there will be no static errors from the drift of the gyroscope. At the same time there will be static errors due to the errors of the accelerometer and magnetometer.

Will accept the control laws in the form

$$W_1(s) = k_P + \frac{k_I}{s} + \frac{k_{2I}}{s^2},\ W_2(s) = r_P + \frac{r_I}{s}. \tag{12}$$

Then expression (11) can be written as follows

$$\left(s^3 I + D_2\right)\boldsymbol{\sigma} = s^2\boldsymbol{\delta}_\omega - \left(k_P s^2 + k_I s + k_{2I}\right)H_{\text{acc}}\boldsymbol{\delta}_{\text{acc}}$$
$$- s\left(r_P s + r_I\right)H_{\text{mag}}\boldsymbol{\delta}_{\text{mag}}, \tag{13}$$

where

$$D_2 = \left(k_P s^2 + k_I s + k_{2I}\right) M_{\text{acc}} + s\left(r_P s + r_I\right) M_{\text{mag}} + s^2 \Omega .$$

We see that there will be no static errors not only from drift of a gyroscope, but also from a magnetometer. Note that when calculating the angles in the coordinate system with a vertical axis, that is, in the coordinate system $OX_oY_oZ_o$, there will be no compensation of the influence of the static error of the magnetometer around the axis OZ_o (as well as the compensation of the effect of the drift of the gyroscope around this axis). This is explained by the fact that in this case when $s = 0$, in the expression for D_2 the third line will be zero.

IV. RESULTS

For calculations we will accept $\omega_x = 0.1\cos 0.3t\ \text{s}^{-1}$, $\omega_y = 0.1\cos 0.5t\ s^{-1}$, $\omega_z = 0.1\cos t\,\text{s}^{-1}$, $\delta_{\omega_x} = 3\cdot 10^{-4}\,\text{s}^{-1}$, $\delta_{\omega_y} = 2\cdot 10^{-4}\,\text{s}^{-1}$, $\delta_{\omega_z} = 1\cdot 10^{-4}\ \text{s}^{-1}$, $\psi_{\text{in}} = 0°$, $\theta_{\text{in}} = 10°$, $\varphi_{\text{in}} = 15°$. The magnetic field is given by vector $[0.3780\ \ 0.5345\ \ -0.7560]^{\text{T}}$. The coefficients of transfer functions (12) are: $k_P = 0.03\ \text{s}^{-1}$, $k_I = 0.001\ \text{s}^{-2}$, $k_{2I} = 0$, $r_P = 0.01\ \text{s}^{-1}$, $r_I = 1\cdot 10^{-4}\ \text{s}^{-2}$.

The disadvantage of using a magnetometer is that its own errors are significant. The estimation of this effect was made by introducing a disturbance [0 0.005 0.005]' to the output signal of magnetometer. We will assume that the coordinate system $OX_rY_rZ_r$ coincides with the initial coordinate system.

Figures 2 and 3 show (using only an accelerometer) the angle errors Δ_ψ, Δ_θ, Δ_φ as the difference between the values of the angles in the presence of drift and angle values in their absence. Figures 2 and 3 correspond to the calculation of the angles in the coordinate systems $OX_oY_oZ_o$ and $OX_rY_rZ_r$ respectively.

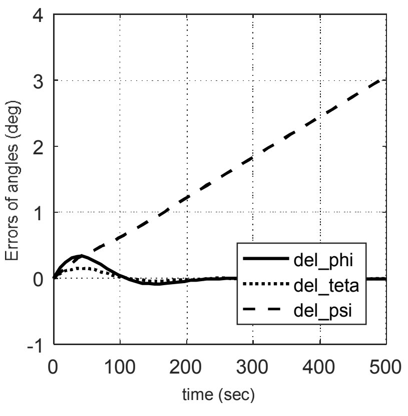

Fig. 2. Errors of angles calculating

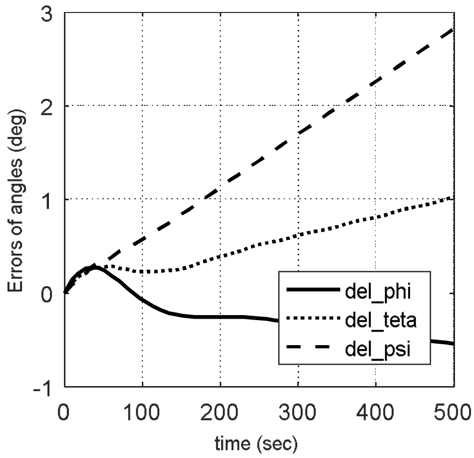

Fig. 3. Errors of angles calculating

When calculating the angles in the coordinate system $OX_oY_oZ_o$ there is a drift relative only to the axis OZ_o.

Figure 4 shows the calculated in coordinate systems $OX_oY_oZ_o$ angular errors using an accelerometer and a magnetometer. We see that when using a magnetometer, the errors are limited in coordinate systems $OX_oY_oZ_o$ (Fig. 5) and $OX_rY_rZ_r$ (Fig. 5).

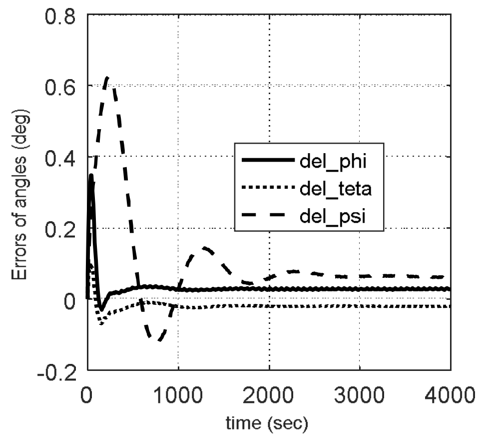

Fig. 4. Errors of angles calculating

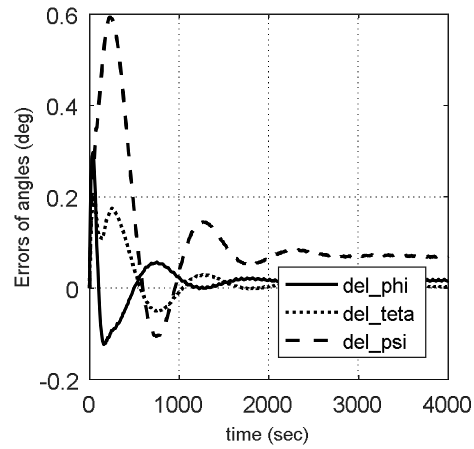

Fig. 5. Errors of angles calculating

To eliminate their influence, the transfer function $W_1(s)$ should be taken as (12). Figures 6 and 7 show the results of calculation in coordinate systems $OX_oY_oZ_o$ and $OX_rY_rZ_r$ respectively, using transfer functions (12). Accepted $k_{2I} = 1 \cdot 10^{-6}$ s^{-3}. In this case there are no static errors in the coordinate system $OX_oY_oZ_o$ relative to the axes OX_o and OY_o. The static error with respect to the axis OZ_o remains, which corresponds to the estimates (13).

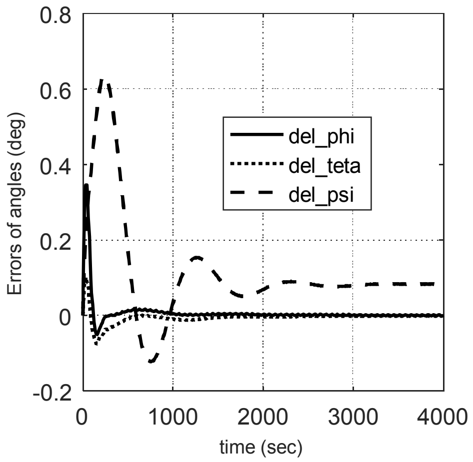

Fig. 6. Errors of angles calculating

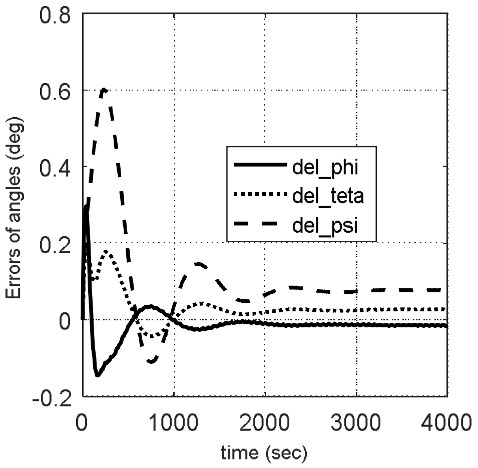

Fig. 7. Errors of angles calculating

In the coordinate system $OX_rY_rZ_r$ (Fig. 7) there are static errors relative to all axes.

V. CONCLUSION

For theoretical analysis, instead of the full equation of the complementary filter, one can use the linearized equation for filter errors. When calculating the angles in an arbitrary reference coordinate system in the case of correction only from the accelerometer there are the errors from the drift of the gyroscope relative to all axes of this coordinate system. Proper choice of the transfer function in the correction channel from the accelerometer can not only eliminate the unlimited increasing of errors due to the drift of the gyroscope, but also eliminate static errors due to the errors of the magnetometer relative to the horizontal axes.

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Л. М. Рижков. Синтез та аналіз комплементарного фільтра для визначення орієнтації рухомого об'єкта

Розглянуто проблеми синтезу та аналізу комплементарних фільтрів, використовуваних під час визначення орієнтації рухомих об'єктів. У статті досліджується комплементарний фільтр в довільно розташованій еталонній системі координат, що визначається специфікою функціонування вимірювального пристрою. Представлено

передавальні функції контролера у каналах корекції за сигналами акселерометра і магнітометра. Показано, що при використанні гіроскопа і акселерометра виникає дрейф відносно усіх осей еталонної системи координат. Отримано співвідношення для розрахунку цих дрейфів. Спрощені рівняння комплементарного фільтра було отримано відносно похибки орієнтації. Проаналізовано вплив структури передавальної функції контролера в каналі корекції за сигналами акселерометра на статичну похибку комплементарного фільтра. Показано, що при перетворенні кутів орієнтації в систему координат з вертикальною віссю можна усунути статичні похибки магнітометра щодо горизонтальних осей, при цьому похибка щодо вертикальної осі зберігатиметься.
Ключові слова: комплементарний фільтр; орієнтація рухомого об’єкта; гіроскоп; акселерометр; магнітометр.

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Л. М. Рыжков. Синтез и анализ комплементарного фильтра для определения ориентации подвижного объекта

Рассмотрены проблемы синтеза и анализа комплементарных фильтров, используемых при определении ориентации подвижных объектов. В статье исследуется комплементарный фильтр в произвольно расположенной отсчетной системе координат, определяемой спецификой функционирования измерительного устройства. Представлены передаточные функции контроллера в каналах коррекции по сигналам акселерометра и магнитометра. Показано, что при использовании гироскопа и акселерометра возникает дрейф относительно всех осей отсчетной системы координат. Получены соотношения для определения этих дрейфов. Упрощенные уравнения комплементарного фильтра были получены относительно ошибки ориентации. Проанализировано влияние структуры передаточной функции контроллера в канале коррекции по сигналам акселерометра на статическую точность комлементарного фильтра. Показано, что при преобразовании углов ориентации в систему координат с вертикальной осью можно устранить статические погрешности магнитометра относительно горизонтальных осей, при этом погрешность относительно вертикальной оси сохраняется.
Ключевые слова: комплементарный фильтр; ориентация подвижного объекта; гироскоп; акселерометр; магнитометр.

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