MATHEMATICAL MODELING OF PROCESSES AND SYSTEMS

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N. M. Glazunov

ON MATHEMATICAL MODELING OF (NANO) TECHNOLOGIES RELATED TO NAVIGATION PROBLEMS ON THE BASE OF GENERALIZATIONS OF THE PICARD METHOD

Department of Electronics, National Aviation University, Kyiv, Ukraine
E-mail: glanm@yahoo.com

***Abstract*—*The aim of the paper is the mathematical modeling of nanotechnology problems of navigation based on generalizations of the Picard method. The Picard method for solving systems of ordinary differential equations, and its extensions on the basis of hyperlogarithms and iterated path integrals, are presented. The derivation of the Picard-Fuchs differential equations for connections in bundles on schemes is given. The results can be used to study the corresponding differential equations and to calculate the Taylor coefficients of (dimensionally regularized) Feynman amplitudes with rational parameters.*

Index Terms—Picard method; ordinary differential equation; iterated integral; hyperlogarithm; nanotechnology; Picard–Fuchs differential equation; multiple zeta value.

I. INTRODUCTION

It is considered the Picard method for solving systems of ordinary differential equations (ODE) and its generalization on the basis of iterative integrals. The proposed methods can be used to solve based on (nano) technologies of selected elements of air navigation problems [1]–[6], [18]. Along with selected navigation problems, it includes some of the tasks of sensors, amplifiers, modulators, lasers.

The proposed methods are based on Picard's method [7]–[9], and are also a far-reaching development of this method.

In the framework the general scheme of Picard's method is presented. After that we recall the Lappo–Danilevskii's matrix method and the extension of the Picard method. The derivation of the Picard–Fuchs differential equations for connections in bundles on schemes is given. Then we introduce iterated path integrals by Parshin, Chen, and others [10], [11]. These integrals can be regarded as a far-reaching development of Picard's method. Next multiple polylogarithms are presented. Multiple zeta values and multiple polylogarithms are applied under Feynman quantization. Such quantization is necessary for the investigation (radiation) of plasmon-polaritons and another quantum effects used in some sensors, modulators and other nano devices. By Broadhust and Kraimer the Tailor coefficients (dimensionally regularized) of Feynman amplitudes with rational parameters are multiple zeta values (see [12]–[14]). Feynman amplitudes give the coefficients of the perturbation theory series.

II. ELEMENTARY PICARD METHOD

Let M be a differentiable manifold and T_M is its tangent bundle. A vector field on M is the mapping $X: M \to T_M$, that satisfies the condition $\pi \circ X = id_M$, where π is the natural projection $T_M \to M$.

Example

Consider the differential equation $\dot{x} = 2x,\ t_0 = 0$.

The Picard method in this simple case has the form: let $x_0(t_0) = 1$,

$$x_1 = 1 + 2\int_0^t d\tau = 1 + 2t,$$

$$x_2 = 1 + 2\int_0^t (1+2\tau)d\tau = 1 + 2t + 2t^2, ,$$

$$x_3 = 1 + 2\int_0^t (1+2\tau+2\tau^2)d\tau = 1 + 2t + 2t^2 + \frac{4t^3}{3},$$

. . . .

$$x_n = 1 + 2t + 2t^2 + \cdots + \frac{2^n t^n}{n!},$$

. . . .

$$\lim_{n\to\infty} x_n = e^{2t}.$$

More generally, let $C: M \to M$ be the contraction mapping of a metric space M. Let $M = R^{n+1}$ and let $\dot{x} = v(t,x),\ x \in M$, be the differential equation, where $v(t,x)$ is the vector field in some domain of the extended phase space R^{n+1}. For n-dimensional vector x from M let $\varphi: t \mapsto x$ and put

$$(C\varphi)(t) = x_0 + \int_{t_0}^{t} v(\tau, \varphi(\tau))\, d\tau.$$

This gives the vector form of the Picard method.

III. Lappo–Danilevskii Matrix Method

Let $x = (x_1, ..., x_n)^{\mathrm{T}}$,

$$\dot{x} = A(t)x, \ x \in R^n,$$

be a system of ordinary differential equations with variable coefficients. Here $A(t)$ is a $n \times n$ matrix of continuous functions defined on an open subset U of the real line. In the framework of the matrix valued solutions and fundamental matrices of the system let $t_0 \in U$ and $X_0 = X(t_0)$ be the initial condition. In matrix notations the differential equation is equivalent to the integral equation

$$X(t) - X(t_0) = \int_{t_0}^{t} A(\tau) X(\tau) d\tau.$$

In this case the Picard method has the form

$$X_{n+1}(t) = X_n(t) + \int_{t_0}^{t} A(\tau) X_n(\tau) d\tau, \ n = 0, 1, \ldots.$$

By the Liouville formula

$$\det X(t) = \exp \int_{t_0}^{t} Sp\, A(\tau)\, d\tau,$$

where $Sp\, A(\tau)$ is the trace of $A(\tau)$.

Put $t_0 = 0$ and let $X(t_0) = E$ where E is the unit matrix. Let now $A(\tau)$ be the matrix with rational coefficients.

Under the solution of the Poincare problem [15] on Fuchsien type differential equations Lappo–Danilevskii [7] has developed the matrix method and introduced hyperlogarithm. By definition hyperlogarithms are functions of the form

$$l(a, z) = \int_0^z \frac{d\tau}{\tau - a},$$

$$l(a_1, \ldots, a_m;\ z) = \int_0^z \frac{d\tau}{\tau - a_1} l(a_2, \ldots, a_m;\ \tau),$$

$$l(a_1, \ldots, a_m; z) = \int_0^z \frac{d\tau_1}{\tau_1 - a_1} \int_0^{\tau_1} \frac{d\tau_2}{\tau_2 - a_2} \cdots \int_0^{\tau_{m-1}} \frac{d\tau_m}{\tau_m - a_m}.$$

Put

$$\omega_0 = \frac{d\tau}{\tau}, \quad \int_0^z \frac{d\tau}{\tau} = \log z,$$

$$\omega_1 = \frac{d\tau}{\tau - 1}, \ \int_0^z \frac{d\tau}{\tau - 1} = \log(1 - z).$$

Then

$$\int_0^z \omega_0 \omega_1 = \int_0^z \frac{\log(1-t)dt}{t}.$$

IV. Picard–Fuchs Differential Equations

An interesting class of differential equations can be constructed from elliptic curves. It is well known that the Weierstrass function $\wp$ and its derivative $\wp'$ satisfy the differential equation

$$(\wp')^2 = 4\wp^3 - 60c_2\wp - 140c_3.$$

A. *Fuchsian type differential equations*

Let K be the field of real or complex numbers, $K(U)$ be the field of meromorhpic functions on U and $H(U)$ be the ring of holomorphic functions on U.

The Fuchsian differential equations has the form

$$y^{(\partial)} + a_1(t) y^{(n-1)} + \cdots + a_n(t) y = 0,$$

where $a_i(t) \in K(U)$. If $t^j a_i(t) \in H(U)$ then it is called the Fuchsian type differential equation.

Below we use elements of scheme theory [13], [16], [6]. In this framework let S/k be a smooth scheme over a field k, U be some element of an open covering S, O_S be a structural sheaf on S, $\Gamma(U, O_S)$ be a section of O_S on U.

For a one-dimensional sheaf of germs of differentials $\Omega^1_{S/k}$ and a coherent sheaf F on S, a connection on a sheaf F is called a homomorphism of sheaves

$$\nabla : F \to \Omega^1_{S/k} \otimes F,$$

such that if $f \in \Gamma(U, O_S)$, $g \in \Gamma(U, F)$, then

$$\nabla(fg) = f\nabla(g) + df \otimes g.$$

Let $f: X \to S$, $S = Spec\, B$, $X = Spec\, A$, be the morphism of smooth affine schemes over k.

Example 1

Consider the elliptic curve $y^2 = x^3 + t$. In this case $B = k[t, t^{-1}]$, $A = B[x, y]/(y^2 - x^3 - t)$. The invariant differential on this curve is of the form $\omega = dx / y$, and, after calculations with differentials, we obtain the Picard–Fuchs differential equation

$\frac{d\omega}{dt}+\frac{1}{6t}\omega=0$, which describes how this invariant differential varies in the family.

Let

$$\dot{X}=A(t)X,$$

be the Fuchsian type differential equation in matrix notations.

It is possible to generalized the previous considerations.

B. Integration of connections

Recall that the cochain complex

$$\left(K^{\bullet},d\right)=\left\{K^{0}\xrightarrow{d}K^{1}\xrightarrow{1}K^{2}\xrightarrow{d}\cdots\right\}$$

in the category of abelian groups is the sequence of abelian groups and morphisms $d:K^{p}\rightarrow K^{p+1}$ such that $d\circ d=0$. Let $\Omega^{i}_{S/k}$ be a sheaf of germs of i-differentials, F the coherent sheaf on S, $\nabla^{i}(\alpha\otimes f)=d\alpha\otimes f+(-1)^{i}\alpha\wedge\nabla(f)$, where $\alpha\in\Omega^{i}_{S/k}$. Then homomorphisms ∇, ∇^{i} define the sequence of homomorphisms

$$F\rightarrow\Omega^{1}_{S/k}\otimes F\rightarrow\Omega^{2}_{S/k}\otimes F\rightarrow\cdots.$$

The connection *is integrable* if the last sequence is a complex.

Proposition. The next conditions a), b) are equivalent: a) the connection ∇ is integrable; b) $\nabla\circ\nabla^{1}=0$.

Example 2

Let $F=O_{S}$ be the structural sheaf on a scheme S. Then

$$\nabla:O_{S}\rightarrow\Omega^{1}_{S/k}\otimes O_{S}\approx\Omega^{1}_{S/k},$$

and so $\nabla(f)=df$. This connection is integrable, since the exterior differentiation operator d defines a de Rham complex:

$$O_{S}\rightarrow\Omega^{1}_{S/k}\otimes O_{S}\rightarrow\Omega^{2}_{S/k}\otimes O_{S}\rightarrow\cdots.$$

Example 3

Let $y^{2}=x(x-1)(x-\lambda)$ be the Legendre family of elliptic curves over the field of complex numbers C with $\lambda\in C-\{0,1\}$. The Picard–Fuchs differential equation

$$\frac{d^{2}}{dt^{2}}\omega+\frac{(2t-1)}{t(t-1)}\frac{d}{dt}\omega+\frac{1}{4t(t-1)}\omega=0,$$

of the family encodes many properties of such a family of elliptic curves.

V. ITERATED PATH INTEGRALS

Here we follow to [10], [11]. Let C be the complex plane and $f_{i}(z)$ be the holomorphic function on C. Let $f(z)dz$ be the differential of the first kind on C. Let S be a Riemann surfaces and w be the differential of the first kind on S. Parshin has considered iterated integrals of this type on Riemann surfaces [10]. Chen [11] in some analogy with [7] for smooth paths on a manifold M and respective path spaces have investigated iterated (path) integrals. For differential forms $\omega_{1},\ldots,\omega_{r}$ on M he has constructed the iterated integrals by repeating r times the integration of the path space differential forms (and their linear combinations). Chen [11] has denoted the iterated integrals as $\int\omega_{1}\bullet\cdots\bullet\omega_{r}$ and set $\int\omega_{1}\bullet\cdots\bullet\omega_{r}=1$ when $r=0$ and $\int\omega_{1}\bullet\cdots\bullet\omega_{r}=0$ when $r<0$. More generally iterated integrals are path space differential forms which permit further integration.

VI. MULTIPLE POLYLOGARITHM

Define polylogarithm

$$Li_{m}(z)=\sum_{n=1}^{\infty}z^{n}n^{-m}.$$

Example 4

$$Li_{2}(1)=\zeta(2)=\frac{\pi^{2}}{6},$$

where $\zeta(s)$ is the Riemann zeta function.

Example 5

In the framework of hyperlogarithms we have:

$$Li_{1}(z)=-\int_{0}^{z}\omega_{1},$$

$$Li_{2}(z)=-\int_{0}^{z}\frac{dt_{1}}{t_{1}}\int_{0}^{t_{1}}\frac{dt_{2}}{1-t_{2}}=-\int_{0}^{z}\omega_{0}\bullet\omega_{1}.$$

Multiple polylogarithm is defined as

$$Li_{n_{1},\ldots,n_{p}}(z_{1},\ldots,z_{p})=\sum_{0<k_{1}<\cdots<k_{p}}\frac{z_{1}^{k_{1}}\cdots z_{p}^{k_{p}}}{k_{1}^{n_{1}}\cdots k_{p}^{n_{p}}}.$$

Example 6

The special value of the multiple polylogarithm is the multiple zeta value of the weight w and the depth p [12], [13], [17]:

$$Li_{n_{1},\ldots,n_{p}}(1,\ldots,1)=\zeta(n_{1},\ldots,n_{p}).$$

Feynman integrals and amplitudes and their representation by multiple polylogarithms and multiple zeta values will be presented elsewhere.

VII. Conclusions

It is given vector and matrix extensions of the Picard method. These extensions are based on the Lappo–Danilevskii`s matrix method and on iterative integrals. Picard–Fuchs differential equations and methods of their representation by connections on algebraic varieties are analyzed. For the applications to Feynman integrals and amplitudes multiple polylogarithms and multiple zeta values are presented. Numerical examples are included.

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Glazunov Nikolaj. Doctor of Phys.-Math. Sciences. Senior researcher. Professor.
Department of Electronics, National Aviation University, Kyiv, Ukraine.
Education: Lvov state university, Lvov, USSR, (1969).
Research interests: arithmetic algebraic geometry and its applications, dynamical systems, optimization.
Publications: 235.
E-mail: glanm@yahoo.com

М. М. Глазунов. Математичне моделювання (нано) технологій пов'язаних з задачами навігації на основі узагальнень метода Пікара

Розглянуто математичне моделювання (нанотехнологічних) задач навігації на основі узагальнень методу Пікара. Представлено метод Пікара для розв'язування систем звичайних диференціальних рівнянь та його розширення на основі гіперлогарифмів та ітерованих інтегралів. Наведено виведення диференціальних рівнянь Пікара–Фукса для зв'язностей у пучках та в розшаруваннях на схемах. Результати можуть бути використані для вивчення відповідних диференціальних рівнянь та для розрахунку коефіцієнтів Тейлора (розмірно регуляризованих) амплітуд Фейнмана з раціональними параметрами.

Ключові слова: метод Пікара; звичайне диференціальне рівняння; ітерований інтеграл; гіперлогарифм; нанотехнології; диференціальне рівняння Пікара–Фукса; кратне значення дзета.

Микола Михайлович Глазунов. Доктор фізико-математичних наук. Старший науковий співробітник. Професор.
Кафедра електроніки, Національный авіаційний університет, Київ, Україна.
Освіта: Львівський державний університет, Львів, СРСР, (1969).
Направлення наукової діяльності: арифметична алгебраїчна геометрія та її застосування, динамічні системи, оптимізація.
Кількість публікацій: 235.
E-mail: glanm@yahoo.com

Н. М. Глазунов. Математическое моделирование (нано) технологий связанных с задачами навигации на основе обобщений метода Пикара

Рассмотрено математическое моделирование (нанотехнологических) задач навигации на основе обобщений метода Пикарда. Приводятся метод Пикара для решения систем обыкновенных дифференциальных уравнений и его расширения на основе гиперлогарифмов и итерированных интегралов. Приводится вывод дифференциальных уравнений Пикара-Фукса для связей в пучках и в расслоениях на схемах. Результаты могут быть использованы для изучения соответствующих дифференциальных уравнений и расчета коэффициентов Тейлора (размерно-регуляризованных) амплитуд Фейнмана с рациональными параметрами.

Ключевые слова: метод Пикара; обыкновенное дифференциальное уравнение; итерированный интеграл; гиперлогарифм; нанотехнологии; дифференциальное уравнение Пикара-Фукса; кратное дзета-значение.

Николай Михайлович Глазунов. Доктор физико-математичних наук. Старший научный сотрудник. Профессор.
Кафедра электроники, Национальный авиационный университет, Киев, Украина.
Образование: Львовский государственный университет, Львов, СССР, (1969).
Направление научной деятельности: арифметическая алгебраическая геометрия и ее приложения, динамические системы, оптимизация.
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E-mail: glanm@yahoo.com