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AN EXAMPLE OF AN ALTERNATIVE METHOD OF THE NORMAL DISTRIBUTION DENSITY DERIVATION VIA A CONCEPT OF A MULTI-OPTIONAL OPTIMALITY

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Abstract**—***Considered a multi-optional method of finding a random value normal probability distribution density. Specific hybrid optional functions are taken into account at the optimization of an objective functional which includes an entropy uncertainty measure for those specific hybrid optional functions. Required mathematical models for obtaining the optimal multi-optional distributions suppose existence of a random value's first and second moments of the distribution density. Normal distribution density is obtained in the way which does not deal with probability derivations but applies a multi-optional optimality concept instead. As a result, it is revealed that normal distribution density is the hybrid multi-optional effectiveness function delivering an extremal value to the objective functional. This is a new insight into the scientific substantiation of the well-known dependency derived in another way; also it is a new explanation of the widely spread in nature phenomenon.*

Index Terms—Normal distribution; distribution density; parameter of distribution; optimization; entropy extremization principle; multi-optionality; hybrid optional function; optimal distribution; variational problem.

I. INTRODUCTION

Normal distribution is widely spread in the nature and scientific research. Whether the research is connected with measuring something, for instance, aircraft noise [1], or dealing with aeronautical engineering maintenance technologies [2], or estimation of quality parameters in the radio flight support operational systems [3], it requires the accuracy of the related evaluations. However, the measuring processes accompanying either inaccuracies or observational errors are indispensable and most often they are assumed to have normal probabilistic distribution density.

Normal distribution is also a crucial point at probabilistic assessments of continuous random values with respect to multiple independent unpredicted disturbances influencing separately infinitesimally the stochastic results applicably, for example, to design works [4] or control functioning modes [5].

Normal distribution is fairly well investigated; there is one more ideological concept's issue although. The point is that development of scientific principles based upon substantiated concepts makes it possible to discover certain new theoretical explanations even to already well-known dependences.

In all mentioned above there is a temptation to use an entropy approach proposed and developed in a series of monographs by professor Kasianov V. A. (an enormously powerfully prolific theoretician) and his follower (both from National Aviation University, Kyiv, Ukraine) [6] – [8]. The cornerstone of the subjective preferences theory (subjective analysis) [6] – [8] is the Subjective Entropy Maximum Principle (SEMP) doctrine, which was further transformed into a hybrid-optional functions distribution optimality concept [9] – [16].

Thus, the purpose of the paper is to give a new impulse to the multi-optional hybrid functions entropy application with respect to the unsolved part of the general problem of the optimum finding concerning normal distribution density [17], [18].

A combined multi-optional approach adopts similar ones initiated in papers [9] – [16].

II. SOLUTION OF THE PROBLEM

A. Traditional Theory Approach

As it is well-known the probability density of a normally distributed continuous random value is given with the formula of [17, Chapter 6, pp. 116–120, Sub-Chapter 6.1, especially P. 117, (6.1.1)], [18, § 3, pp. 52–75, esp. p. 3.2, P. 60]:

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-a)^2}{2\sigma^2}}, \tag{1}$$

where $f(x)$ is the density of the normal distribution; x is the continuous random value; σ is the mean quadratic deviation value; $\pi \approx 3.14$; a is the continuous random value x expectation.

One way of (1) derivation is as follows, [18, § 3, pp. 52–75, esp. p.p. 3.1, 3.2, pp. 52–60]. We present the derivation's concise version with emphasizing the principal moments.

Consider some point $M(x, y)$ on the coordinate plane (x, y) and a negligibly small rectangular square area of $\Delta s = \Delta x \Delta y$ with the dimensions of Δx and Δy along the coordinate axes' around the point. The probability P of a random hitting the area Δs on condition of the coordinate strips' Δx and Δy hitting independency is

$$P = P(\Delta x)P(\Delta y) \approx f(x)f(y)\Delta s, \quad (2)$$

where $P(\Delta x)$ and $P(\Delta y)$ are the corresponding probabilities of hitting the related strips of Δx and Δy; $f(x)$ and $f(y)$ are the corresponding probability density distributions [18, pp. 53–55].

The same speculations for any other rectangular coordinate system with the same origin Ox_1y_1 give the similar to (2) result, [18, P. 55]:

$$P_1 \approx f(x_1)f(y_1)\Delta s_1, \quad (3)$$

where x_1 and y_1 are the new coordinates of the point M; Δs_1 is the square area of the new rectangular.

Choosing $\Delta s = \Delta s_1 \to 0$, we come to the main equation, [18, P. 55]:

$$f(x)f(y) = f(x_1)f(y_1). \quad (4)$$

Arbitrary choosing the axis Ox_1 aiming at the point M, differentiating the obtained equations with respect to x and y, it yields [18, pp. 55, 56]:

$$f(x)\frac{df(y)}{dy}\frac{\sqrt{x^2+y^2}}{y} = \frac{df(x)}{dx}f(y)\frac{\sqrt{x^2+y^2}}{x}. \quad (5)$$

$$\frac{df(y)}{dy}\frac{1}{yf(y)} = \frac{df(x)}{dx}\frac{1}{xf(x)}. \quad (6)$$

$$\frac{1}{xf(x)}\frac{df(x)}{dx} = b, \qquad \frac{df(x)}{f(x)} = bxdx. \quad (7)$$

$$\ln f(x) = \frac{bx^2}{2} + \ln C, \qquad f(x) = Ce^{\frac{bx^2}{2}}. \quad (8)$$

Since, [18, P. 57]:

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} Ce^{-\frac{h^2x^2}{2}}dx = 1, \quad (9)$$

where $b = -h^2$, we have, [18, P. 57]:

$$C = \frac{1}{\int_{-\infty}^{\infty} e^{-\frac{h^2x^2}{2}}dx} = \frac{h}{\sqrt{2\pi}}. \quad (10)$$

$$f(x) = \frac{h}{\sqrt{2\pi}}e^{-\frac{h^2x^2}{2}}. \quad (11)$$

If the target is not at the origin of the coordinate system, but at some point along the Ox axis displaced on a distance of a, one has to substitute x for $x - a$ in (11), [18, P. 57]:

$$f(x) = \frac{h}{\sqrt{2\pi}}e^{-\frac{h^2(x-a)^2}{2}}. \quad (12)$$

With taking into account that accuracy constant, [18, pp. 57, 59],

$$h = \frac{1}{\sigma} \quad (13)$$

we obtain (1).

B. Multi-Optional Concept

On the other hand one can present the process of random points' distribution along the axis as a multi-optional problem. The things to be taken into consideration in this case are: 1) "optionality" of the quadratic values of $x - a$: $(x-a)^2$; with 2) taking into account the quadratic value of the $x - a$ optional distribution accuracy h: h^2; and 3) uncertainty of supposed random value x probability distribution density $f(x)$.

The most important here is to understand that there must be some optimality in the framework of the nature things "optionality". The approach similar to seeking after preferences in subjective analysis [6] – [8], and applied to hybrid optional optimal distribution densities findings [9] – [16], allows implementing the objective functional of the following kind:

$$G_f = \int_{-\infty}^{\infty}\left[-f(x)\ln f(x) - \frac{1}{2D}f(x)(x-m_x)^2\right]dx + \gamma\left[\int_{-\infty}^{\infty} f(x)dx - 1\right] - \ln \Delta x, \quad (14)$$

where D is the dispersion, here we imply that accuracy h relates with dispersion D via equality (13) and $D = \sigma^2$; m_x is the expectation of the random value x; γ is the internal structural parameter of the hybrid optional distribution function $f(x)$ (random value x probability distribution density) as an uncertain Lagrange multiplier for the normalizing condition (9), together $\frac{1}{2D}$ and γ are analogous to the parameters characterizing a system's hybrid optimal optional behavior [9] – [16], likewise for the active element's psych [6] – [8] (endogenous parameter for the function of the optional effectiveness $(x-m_x)^2$ and uncertain Lagrange multiplier for the normalizing condition $\int_{-\infty}^{\infty} f(x)dx - 1$ respectively); Δx is the degree of accuracy at the hybrid optional function distribution density entropy (analogous to the subjective entropy of the preferences) determination [17, pp. 493-502, Sub-Chapter 18.7, esp. P. 495, (18.7.2)-(18.7.4)].

Thus, we propose to use an optimization method which resembles SEMP of subjective analysis, but the proposed method differs absolutely from SEMP [6] – [8], since, being applied for a continuous optional value $x-m_x$, the method does not imply or consider any of active system's elements at all [9] – [16]. Only objectively existing characteristics of a continuous random value probability distribution density, however, presupposed with the background of the density of the probability distribution uncertainty are utilized.

The first integral member of the objective functional (14) is the exact distribution uncertainty parameter in the view of the distribution's optional function's entropy like also discussed at [17, pp. 493-502, Sub-Chapter 18.7, esp. P. 495, (18.7.2)-(18.7.4)].

The multiplier of $1/2$ in functional (14) implies symmetrical quadratic accuracy h^2 of the quadratic random value distribution $(x-m_x)^2$ with respect to the distribution's expectation m_x; hence, it is divided into halves. The sign "minus" in front of the hybrid optional effectiveness function: $\frac{1}{2D} f(x)(x-m_x)^2$ (the second integral member of the objective functional (14)) means the existence of relatively higher density distribution $f(x)$ values in areas pertaining with lower optional effectiveness function: $(x-m_x)^2$.

The necessary conditions of functional (14) extremum existence in the view of the well-known Euler-Lagrange equation, [19, Chapter I, § 4, pp. 20-28, esp. P. 21, (4)]:

$$\frac{\partial F^*}{\partial f(x)} - \frac{d}{dx}\left(\frac{\partial F^*}{\partial f'(x)}\right) = 0 \quad (15)$$

where F^* is the underintegral function of the integral of (14); $f'(x)$ is the first derivative of the sought after probability density distribution function of $f(x)$ with respect to x, yield, since

$$\frac{\partial F^*}{\partial f'(x)} \equiv 0 \quad \text{and} \quad \frac{\partial F^*}{\partial f(x)} = 0, \quad (16)$$

$$\frac{\partial F^*}{\partial f(x)} = -\ln f(x) - 1 - \frac{1}{2D}(x-m_x)^2 + \gamma = 0. \quad (17)$$

Then

$$\ln f(x) = \gamma - 1 - \frac{1}{2D}(x-m_x)^2. \quad (18)$$

$$f(x) = e^{\gamma - 1 - \frac{1}{2D}(x-m_x)^2} = e^{\gamma-1} e^{-\frac{1}{2D}(x-m_x)^2}. \quad (19)$$

With the use of the normalizing condition (9)

$$\int_{-\infty}^{\infty} f(x)dx = 1 = \int_{-\infty}^{\infty} e^{\gamma-1} e^{-\frac{1}{2D}(x-m_x)^2} dx. \quad (20)$$

$$e^{\gamma-1} = \frac{1}{\int_{-\infty}^{\infty} e^{-\frac{1}{2D}(x-m_x)^2} dx}. \quad (21)$$

Therefore

$$f(x) = \frac{1}{\int_{-\infty}^{\infty} e^{-\frac{1}{2D}(x-m_x)^2} dx} e^{-\frac{1}{2D}(x-m_x)^2}. \quad (22)$$

The integral in the denominator is the well-known Euler's-Poisson's integral [17, Chapter 6, pp. 116-120, Sub-Chapter 6.1, esp. P. 117, (6.1.2), (6.1.3)]. After the substitution of the independent variable of the integration

$$\frac{x-m_x}{\sqrt{2D}} = t, \quad (23)$$

we have likewise (10)

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2D}(x-m_x)^2} dx = \sqrt{2D}\sqrt{\pi}. \quad (24)$$

$$f(x) = \frac{1}{\sqrt{D}\sqrt{2\pi}} e^{-\frac{1}{2D}(x-m_x)^2}. \quad (25)$$

At last (1) again, but obtained in the different, (14) – (25), from the probabilistic way of derivation (2) – (13).

III. Conclusions

Proposed approach engaging an uncertainty measure in type of entropy, applied for distribution density hybrid optional functions optimization, allows finding normal distribution density, without probabilities determination, in a new multi-optional way. The accepted suppositions are the spreading of a random value having its expectation and dispersion (accuracy) of the value's distribution; as well as existence of the distribution density uncertainty suspected in delivering an extremal value to some objective functional.

As a result, it is revealed that normal distribution density is optimal for an objective functional including the distribution's density entropy, as well as taking into account, with the accuracy reversibly proportional to doubled dispersion, the higher probability density for smaller deviations of the random value from its expectation. This is so obvious in widely spread normal distribution.

Such approach and interpretations broaden the horizons of scientific explanations for occurring normal distribution optimality; and it encourages further research in the field of hybrid optional functions optimal distributions.

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А. В. Гончаренко. Приклад альтернативного методу виводу щільності нормального розподілу через концепцію оптимальності багатоопційності

Розглянуто багатоопційний метод знаходження щільності нормального розподілу ймовірності випадкової величини. Специфічні гібридні опційні функції взято до уваги при оптимізації цільового функціоналу, котрий включає ентропійну міру невизначеності для тих специфічних гібридних опційних функцій. Потрібні математичні моделі для отримання оптимальних багатоопційних розподілів містять припущення про існування першого та другого моментів щільності розподілу випадкової величини. Щільність нормального розподілу отримується у такий спосіб, що не має справи із виведенням ймовірності, але застосовує натомість концепцію багатоопційної оптимальності. В результаті, виявляється, що щільність нормального розподілу є тією гібридною багатоопційною функцією ефективності, яка доставляє екстремальне значення даному цільовому функціоналу. Це є новим поглядом на наукове обґрунтування добре знаної залежності, виведеної в інший спосіб; також це є новим поясненням дуже поширеного природного явища.
Ключові слова: нормальний розподіл; щільність розподілу; параметр розподілу; оптимізація; принцип екстремізації ентропії; багатоопційність; гібридна опційна функція; оптимальний розподіл; варіаційна задача.

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А. В. Гончаренко. Пример альтернативного метода вывода плотности нормального распределения через концепцию оптимальности многоопционности

Рассмотрен многоопционный метод нахождения плотности нормального распределения вероятности случайной величины. Специфические гибридные опционные функции приняты ко вниманию при оптимизации целевого функционала, который включает энтропийную меру неопределенности для этих специфических гибридных опционных функций. Требуемые математические модели для получения оптимальных многоопционных распределений содержат допущение о существовании первого и второго моментов плотности распределения случайной величины. Плотность нормального распределения получается способом не имеющим дела с выведением вероятности, но применяющим вместо этого концепцию многоопционной оптимальности. В результате, выявляется, что плотность нормального распределения является той гибридной многоопционной функцией эффективности, которая доставляет экстремальное значение данному целевому функционалу. Это является новым взглядом на научное обоснование хорошо известной зависимости, выведенной другим способом; также это является новым объяснением широко распространенного естественного явления.
Ключевые слова: нормальное распределение; плотность распределения; параметр распределения; оптимизация; принцип экстремизации энтропии; многоопционность; гибридная опционная функция; оптимальное распределение; вариационная задача.

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