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ATTITUDE DETERMINATION BASED ON AXIS AND ANGLE ROTATION COMPUTING

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Abstract**—***The algorithm of attitude determination on the base of axis and angle rotation computing is suggested. Was assumed the known the projections of the normalized vectors in the reference and in the body coordinate systems. The problem is to determine the unit vector and the angle of rotation of the coordinate system connected to the body relative to the reference coordinate system. Comparison with algorithm QUEST is fulfilled. It was shown that the accuracy of proposed algorithm is equivalent to the accuracy of algorithm QUEST.*

Index Terms—Attitude; determination.

I. Introduction

The problem of attitude determination on the basis of information about non-parallel vectors is analyzed. The use of geometric relationships is an effective means of rigid body orientation determination based on the measurement of vectors [1] – [3]. Their characteristic feature is that you do not need to use a matrix (as in the TRIAD [4], [5] algorithm) or quaternion (as in the QUEST [6], [7] algorithm) algebras. The proposed algorithm uses geometric relations, which take place when the body rotates.

II. Problem Statement

Developing the approach proposed in [1], [2], we consider the problem of determining the orientation of a body on the basis of the use of geometric relations. We will assume the known projection of normalized vectors in the reference ($\vec{r}_{oi}$) and in the body ($\vec{r}_i$) coordinate systems. The problem is to determine the unit vector and the angle of rotation of the coordinate system connected to the body relative to the reference coordinate system. That is, the vectors are fixed, their projections are variables in the two specified coordinate systems.

Change the problem statement. We assume that the variables are vectors, that is, we assume that the vectors rotate around the axis of rotation (Fig. 1) in the opposite direction with respect to the direction of rotation of the body-connected coordinate system relative to the reference coordinate system. Trajectories of the ends of vectors are circles with centers on the axis of rotation.

Unlike most known algorithms, we will separately determine the axis of rotation, the angle of rotation and the direction of rotation [1], [2].

III. Problem Solution

In Figure 1 the angle of rotation σ of the plane, which contains the axis of rotation and the vector $\vec{r}_i$, is indicated. A unite vector $\vec{e}$ characterizes the direction of rotation of this plane. The unite vector of rotation of a body $\vec{e}_b$ is equal $\vec{e}_b = -\vec{e}$.

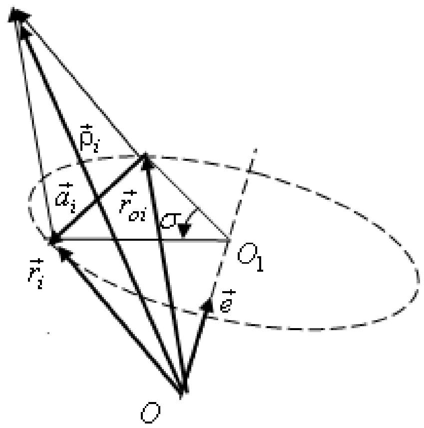

Fig. 1. Vectors

The basis of the algorithm is the use of vectors $\vec{a}_i = \vec{r}_i - \vec{r}_{oi}$, which are in parallel planes and are perpendicular to the vector $\vec{e}$. These vectors correspond to relations

$$\boldsymbol{a}_i^{\mathrm{T}}\mathbf{e} = 0 . \qquad (i = 1, ..., n). \qquad (1)$$

We use this formula to find the vector $\mathbf{e}$. It is essential that from this formula we can find a vector $\mathbf{e}_*$ that is directed along the axis of rotation, but which does not determine the direction of rotation, that is $\mathbf{e}_* = \pm\mathbf{e}$. Therefore, the definition of the direction of rotation must be performed additionally.

Taking into account the errors of measurement of vectors, the problem will be solved on the basis of the method of weighted least squares, that is, the vector $\mathbf{e}_*$ will be determined from the condition of minimizing the loss function

$$l(\mathbf{e}_*) = \frac{1}{2}\sum_{i=1}^{n}\mu_i \left\|\boldsymbol{a}_i^{\mathrm{T}}\mathbf{e}_*\right\|^2 = \frac{1}{2}\sum_{i=1}^{n}\left\|\tilde{\boldsymbol{a}}_i^{\mathrm{T}}\mathbf{e}_*\right\|^2$$
$$= \frac{1}{2}\sum_{i=1}^{n}\left(\tilde{\boldsymbol{a}}_i^{\mathrm{T}}\mathbf{e}_*\right)^{\mathrm{T}}\left(\tilde{\boldsymbol{a}}_i^{\mathrm{T}}\mathbf{e}_*\right) = \frac{1}{2}\sum_{i=1}^{n}\left(\mathbf{e}_*^{\mathrm{T}}\tilde{\boldsymbol{a}}_i\right)\left(\tilde{\boldsymbol{a}}_i^{\mathrm{T}}\boldsymbol{e}_*\right) = \frac{1}{2}\boldsymbol{e}_*^{\mathrm{T}} G \boldsymbol{e}_*, \tag{2}$$

where $\tilde{\boldsymbol{a}}_i = \sqrt{\mu_i}\,\boldsymbol{a}_i$; $G = \sum_{i=1}^{3}\tilde{\boldsymbol{a}}_i\tilde{\boldsymbol{a}}_i^{\mathrm{T}}$; μ_i are non-negative weights.

Consider the condition that the vector $\mathbf{e}_*$ must be unite and assume such a loss function $l_1(\mathbf{e}_*)$:

$$l_1(\mathbf{e}_*) = \frac{1}{2}\mathbf{e}_*^{\mathrm{T}} G \mathbf{e}_* - \lambda\left(\mathbf{e}_*^{\mathrm{T}}\mathbf{e}_* - 1\right), \tag{3}$$

where λ is the Lagrange multiplier.

Consider the problem of minimizing the loss function. Let's write down

$$\frac{\partial l_1(\mathbf{e}_*)}{\partial \mathbf{e}_*} = G\mathbf{e}_* - \lambda\mathbf{e}_* = 0. \tag{4}$$

That is, the condition of the minimum is

$$G\mathbf{e}_* = \lambda\mathbf{e}_*. \tag{5}$$

Then you can write

$$l(\mathbf{e}_*) = \frac{1}{2}\mathbf{e}_*^{\mathrm{T}} G\mathbf{e}_* = \frac{1}{2}\mathbf{e}_*^{\mathrm{T}}\lambda\mathbf{e}_* = \frac{1}{2}\lambda. \tag{6}$$

That is, we are interested in the minimum value of the parameter λ. Thus, the minimization problem is equivalent to finding a vector $\mathbf{e}_*$ as an eigenvector of a matrix, which corresponds to the minimum eigenvalue of the matrix. In the Matlab environment, the eigenvalues of a matrix can be found by function $\mathrm{eig}(G)$. In the absence of measurement errors, we have $G\mathbf{e}_* = 0$, that is, $\lambda_m = 0$. Therefore, for small errors, measurements of vectors can be assumed $\lambda \approx 0$. In this case, the eigenvalue of matrix can be found by such technique.

The eigenvalues of the matrix are the roots of the characteristic equation

$$\Delta = G - I\lambda = -\lambda^3 + f_1\lambda^2 - f_2\lambda + f_3 = 0.$$

Given that we are interested $\lambda \approx 0$, we will accept

$$\Delta \approx -f_2\lambda_m + f_3 = 0.$$

Then

$$\lambda_m \approx \frac{f_3}{f_2}, \tag{7}$$

where $f_2 = \frac{1}{2}\left[\mathrm{tr}\left(G^2\right) - \mathrm{tr}^2\left(G\right)\right]$; $f_3 = \det(G)$.

Next you can find a vector $\mathbf{e}_*$ as solution of equation

$$\left(G - I\lambda_m\right)\mathbf{e}_* = 0.$$

Consider the question of finding the angle of rotation σ and determining the vector $\mathbf{e}$. To do this, we decompose the vectors $\vec{r}_{oi}$ and $\vec{r}_i$ into two mutually perpendicular components (Fig. 2).

$$\begin{aligned} \boldsymbol{u}_{oi} &= \boldsymbol{u}_i = \mathbf{e}_*(\mathbf{e}_*^{\mathrm{T}}\boldsymbol{r}_{oi}) = E\boldsymbol{r}_{oi}, \\ \boldsymbol{m}_{oi} &= \boldsymbol{r}_{oi} - \boldsymbol{u}_{oi} = E_1\boldsymbol{r}_{oi}, \\ \boldsymbol{m}_i &= E_1\boldsymbol{r}_i, \end{aligned} \tag{8}$$

where $E = \mathbf{e}_*\mathbf{e}_*^{\mathrm{T}} = \mathbf{e}\mathbf{e}^{\mathrm{T}}$; $E_1 = I - E$.

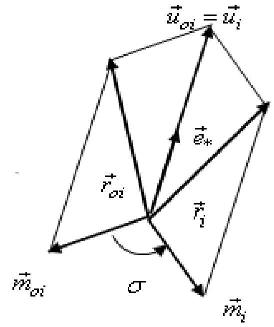

Fig. 2. Angle or rotation

Note that there is a place the relations

$$E = I + E_\times^2; \quad E_1 = -E_\times^2,$$

where

$$E_\times = \begin{bmatrix} 0 & -e_3 & e_2 \\ e_3 & 0 & -e_1 \\ -e_2 & e_1 & 0 \end{bmatrix}; \quad \mathbf{e} = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix}^{\mathrm{T}}.$$

Next we will find

$$\begin{aligned} \boldsymbol{m}_i^{\mathrm{T}}\boldsymbol{m}_{oi} &= \left(E_1\boldsymbol{r}_i\right)^{\mathrm{T}} E_1\boldsymbol{r}_{oin} = \boldsymbol{r}_i^{\mathrm{T}} E_1^{\mathrm{T}} E_1\boldsymbol{r}_{oi} \\ &= \boldsymbol{r}_i^{\mathrm{T}}\left(I - 2E + \boldsymbol{e}\boldsymbol{e}^{\mathrm{T}}\right)\boldsymbol{r}_{oi} = \gamma_i, \end{aligned} \tag{9}$$

where $\gamma_i = \boldsymbol{r}_i^{\mathrm{T}} E_1\boldsymbol{r}_{oi}$.

Rewrite this expression

$$\gamma_i = \gamma_{oi}\cos\sigma, \tag{10}$$

where $\gamma_{oi} = \boldsymbol{r}_{oi}^{\mathrm{T}} E_1\boldsymbol{r}_{oi}$.

Use least-square estimation and specify the lost function

$$f=\sum_{i-1}^{n}\left(\gamma_i-\gamma_{oi}\mu\right)^2, \qquad (11)$$

where $\mu=\cos\sigma$.

Using the condition $\frac{\partial f}{\partial\mu}=0$ we find

$$\mu=\cos\sigma=\sum_{i=1}^{n}\frac{\gamma_i\gamma_{oi}}{\gamma_{oi}^2}. \qquad (12)$$

Let's consider in more detail the choice of vectors $\vec{r}_{oi}$ in order to further find the vector $\vec{e}$. The proposed technique is based on the fact that the vector $\vec{e}$ is perpendicular to the plane in which the vectors $\vec{a}_i$ are located. This means that at least two vectors $\vec{a}_i$ must be non-parallel (in this case, they will form this plane). From Fig. 2 we see that this plane is formed by the components $\vec{m}_{oi}$ and $\vec{m}_i$ of these vectors. Let's illustrate these components for two vectors (Fig. 3).

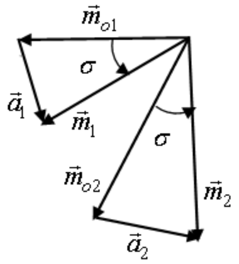

Fig. 3. Analysis of vectors

From this figure, we see that the vectors $\vec{a}_1$ and $\vec{a}_2$ will be parallel when the vectors $\vec{m}_{o1}$ and $\vec{m}_{o2}$ will be in one line. It can also be formulated as follows: the vectors $\vec{a}_1$ and $\vec{a}_2$ will be parallel when the vectors $\vec{r}_{o1}$, $\vec{r}_{o2}$ and the axis of rotation will be in the one plane. Therefore, it is advisable to enter an additional vector $\vec{r}_{o3}=\vec{r}_{o1}\times\vec{r}_{o2}$ and consider the system, at least, of three vectors $\vec{r}_{o1}$, $\vec{r}_{o2}$, $\vec{r}_{o3}$. In this case, you can always form a plane in which there are vectors $\vec{a}_i$.

This result can also be obtained algebraically. Assume that the vectors $\vec{a}_1=\vec{r}_1-\vec{r}_{o1}$ and $\vec{a}_2=\vec{r}_2-\vec{r}_{o2}$ are parallel. Then you can write $\vec{r}_2-\vec{r}_{o2}=k\left(\vec{r}_1-\vec{r}_{o1}\right)$, where k the number. We use the matrix of the directional cosines R and write $\left(R-I\right)\boldsymbol{r}_{o2}=k\left(R-I\right)\boldsymbol{r}_{o1}$, that is, $D\boldsymbol{z}=0$, where $D=R-I$; $\boldsymbol{z}=\boldsymbol{r}_{o2}-k\boldsymbol{r}_{o1}$.

Let's rewrite this expression in the form $D\boldsymbol{z}=\lambda_1\boldsymbol{z}$, where $\lambda_1\to 0$. Then the vector $\boldsymbol{z}$ can be considered as an eigenvector $\boldsymbol{z}_{\lambda_1=0}$ of the matrix $\mathbf{D}$, which corresponds to eigenvalue $\lambda_1=0$. Given that the eigenvector of the matrix $\mathbf{D}$ is equal to the eigenvector of the matrix $\mathbf{R}$, we write $\boldsymbol{z}_{\lambda_1=0}=v\vec{e}$ (v is the number). Then $\vec{r}_{o2}=k\vec{r}_{o1}+v\vec{e}$. This means that the vectors $\vec{r}_{o1}$, $\vec{r}_{o2}$ and the axis of rotation are in the one plane.

The last step in solving the problem is to determine the direction of rotation, considering that in the first stage we found not a vector $\mathbf{e}$ but a vector $\mathbf{e}_*$. As can be seen from Fig. 1, the vector $\vec{a}_i$ coincides in the direction with the cross product $\vec{v}_i=\vec{e}\times\vec{\rho}_i=\vec{e}\times\left(\vec{r}_{oi}+\vec{r}_i\right)$. Given that one of the vectors $\vec{a}_i$ can be zero (in the case when the corresponding vector $\vec{r}_i$ coincides with the axis of rotation), the direction of the vector $\vec{e}$ will be determined in such way

$$\vec{e}=zn\,\vec{e}_*, \qquad (13)$$

where $zn=\operatorname{sign}\left(\vec{v}_1\cdot\vec{a}_1+\vec{v}_2\cdot\vec{a}_2+\vec{v}_3\cdot\vec{a}_3\right)$.

Note that in this case $\sigma\neq\pi$, the direction of rotation can be found in another way, namely, through a function $\sin\sigma$ sign. The positive value of the angle σ corresponds $\boldsymbol{e}=\boldsymbol{e}_*$, and the negative value of the angle σ corresponds $\boldsymbol{e}=-\boldsymbol{e}_*$, that is

$$\boldsymbol{e}=\operatorname{sign}(\sin\sigma)\,\boldsymbol{e}_*. \qquad (14)$$

Using Fig. 2 write

$$\vec{e}_*\sin\sigma=\vec{m}_{o1n}\times\vec{m}_{1n}=M_{o1n}\boldsymbol{m}_{1n}, \qquad (15)$$

where

$$M_{o1n}=\begin{bmatrix}0 & -m_{o1nz} & m_{o1ny}\\ m_{o1nz} & 0 & -m_{o1nx}\\ -m_{o1ny} & m_{o1nx} & 0\end{bmatrix};$$

$\vec{m}_{o1n}$, $\vec{m}_{1n}$ are normalized vectors.

The angle σ will be sought from the condition of minimizing the loss function

$$\rho(\sigma)=\frac{1}{2}\sum_{i=1}^{3}\mu_i\left\|\boldsymbol{e}_*\sin\sigma-M_{o1n}\boldsymbol{m}_{1n}\right\|^2$$

$$=\frac{1}{2}\sum_{i=1}^{3}\mu_i\left(\boldsymbol{e}_*\sin\sigma-M_{o1n}\boldsymbol{m}_{1n}\right)^{\mathrm{T}}\left(\boldsymbol{e}_*\sin\sigma-M_{o1n}\boldsymbol{m}_{1n}\right)$$

$$=\frac{1}{2}\sin^2\sigma\sum_{i=1}^{3}\mu_i-\sin\sigma\sum_{i=1}^{3}\boldsymbol{e}_*^{\mathrm{T}}\tilde{M}_{o1n}\tilde{\boldsymbol{m}}_{1n}$$

$$+\frac{1}{2}\sum_{i=1}^{3}\tilde{\boldsymbol{m}}_{1n}^{\mathrm{T}}\tilde{M}_{o1n}^{\mathrm{T}}\tilde{M}_{o1n}\tilde{\boldsymbol{m}}_{1n}, \qquad (16)$$

where $\tilde{\boldsymbol{m}}_1 = \sqrt{\mu_i}\,\boldsymbol{m}_1$; $\tilde{M}_{o1} = \sqrt{\mu_i}\,M_{o1}$.

To minimize the function of losses we find $\frac{\partial\rho(\sigma)}{\partial\sigma} = \sin\sigma\sum_{i=1}^{3}\mu_i - \boldsymbol{e}_*^T\sum_{i=1}^{3}\tilde{M}_{o1n}\tilde{\boldsymbol{m}}_{1n} = 0$. That is,

$$\operatorname{sign}(\sin\sigma) = \operatorname{sign}\left(\boldsymbol{e}_*^{\mathrm{T}}\sum_{i=1}^{3}\tilde{M}_{o1n}\tilde{\boldsymbol{m}}_{1n}\right). \qquad (17)$$

In the case $\sigma = \pi$ vectors $\boldsymbol{e}$ and $\boldsymbol{e}_*$ are equivalent.

Quaternion of turn can be defined as

$$q = \cos\frac{\sigma_1}{2} + \boldsymbol{e}\sin\frac{\sigma_1}{2},$$

where $\sigma_1 = |\sigma|$; $\sigma = \arccos\left(\sum_{i=1}^{n}\frac{\gamma_i\gamma_{oi}}{\gamma_{oi}^2}\right)$.

Essentially, the proposed algorithm functions for any angle of rotation.

For the three vectors selected in this way, we will analyze the influence of the measurement error of the vector $\vec{r}_2$ on the accuracy of the orientation determination and compare it with the accuracy of the QUEST algorithm.

The vector of error, by a size 0.005, will form perpendicular to the vector $\vec{r}_2$ with variable direction with a step 10° in a plane, perpendicular to the vector $\vec{r}_2$. The new value of vector $\vec{r}_2$ is normalized.

For calculations we will accept $\psi = 60°$; $\theta = 20°$; $\varphi = 10°$; $\mu_i = 1$.

$$\boldsymbol{r}_{1o} = [0 \;\; 0.5547 \;\; 0.8321]',$$
$$\boldsymbol{r}_{2o} = [0.9950 \;\; 0 \;\; 0.0995]'.$$

The results of the calculations for the proposed algorithm and QUEST algorithm are presented in Figs 4 and 5 respectively.

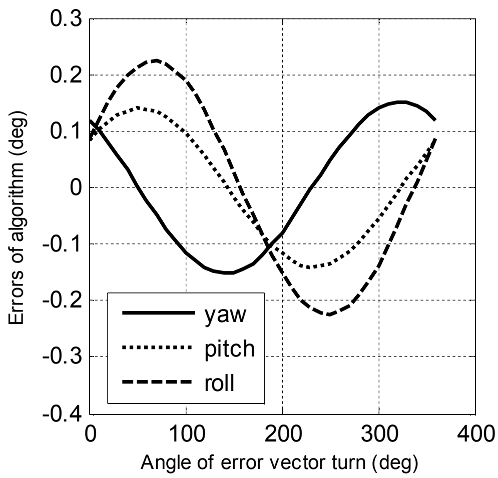

Fig. 4. Errors of angles determination

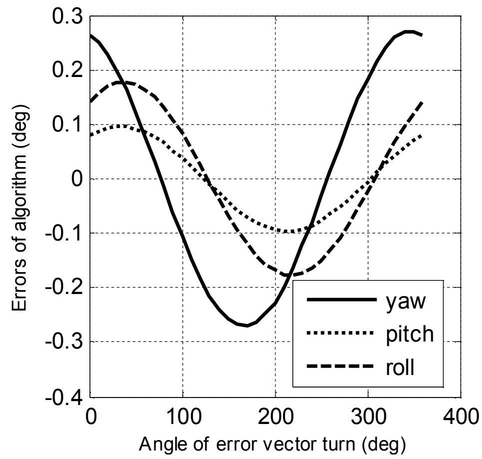

Fig. 5. Errors of angles determination

In Figures 6 and 7 are represented results of computations for another angles and vectors

$$\psi = -60°;\ \theta = 60°;\ \varphi = -40°,$$

$$\vec{r}_{o1} = [-0.8570 \;\; 0.0343 \;\; 0.5142]',$$

$$\vec{r}_{o2} = [0.7053 \;\; 0.7053 \;\; 0.0705]'.$$

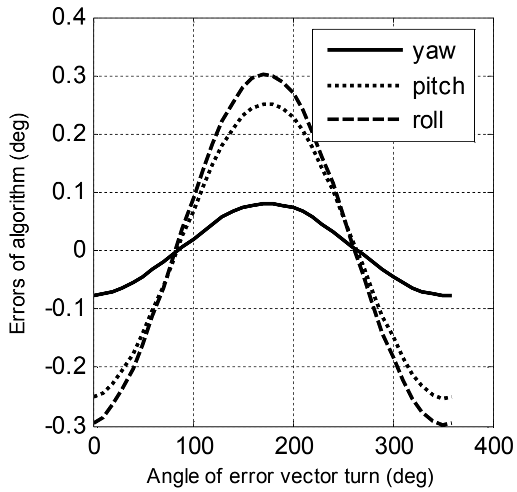

Fig. 6. Errors of angles determination

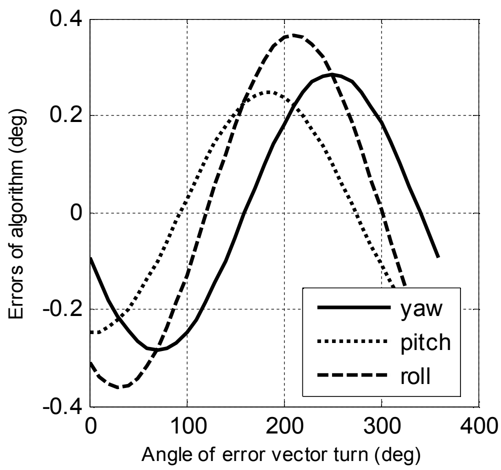

Fig. 7. Errors of angles determination

We see that the proposed algorithm and QUEST algorithm are precisely equivalent.

IV. Conclusions

The use of geometric relationships is an effective means of rigid body orientation determination based. Their characteristic feature is simplicity and visibility. The proposed algorithm functions for any angle of a body rotation. The accuracy of proposed algorithm is equivalent to the accuracy of algorithm QUEST.

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Л. М. Рижков. Визначення орієнтації на базі обчислення осі та кута повороту

Запропоновано алгоритм визначення орієнтації на базі обчислення осі та кута повороту. Вважаються відомими проекції нормованих векторів в опорній та зв'язаній з тілом системах координат. Задача полягає в знаходженні одиничного вектора та кута повороту зв'язаної з тілом системи координат відносно опорної системи координат. Виконано порівняння з алгоритмом QUEST. Показано, що точність запропонованого алгоритму еквівалентна точності алгоритму QUEST.

Ключові слова: орієнтація; визначення.

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Л. М. Рыжков. Определение ориентации на основе вычисления оси и угла поворота

Предложен алгоритм определения ориентации на основе вычисления оси и угла поворота. Считаются известными проекции нормированных векторов в опорной и в связанной с телом системах координат. Задача состоит в определении единичного вектора и оси поворота связанной системы координат относительно опорной системы координат. Выполнено сравнение с алгоритмом QUEST. Показано, что точность предложенного алгоритма эквивалентна точности алгоритма QUEST.

Ключевые слова: ориентация; определение.

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