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# NONCLASSICAL QUATERNIONS AND PENTANIONS IN PROBLEMS OF INERTIAL ORIENTATION

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***Abstract*—*The article considers the nonclassical quaternions and pentanions of helf-rotations of solid body and their application in problems of control and orientation of moving objects. In contrast to classical rationed Hamiltonian quaternions of complete rotations the nonclassical quaternions of helf-rotations may be null, they have variable rates, depending on the angle of Euler finite rotation.***

**Index Terms**—Non-classical quaternions, pentanions of helf-rotations; strapdown inertial orientation systems, orientation control.

## I. INTRODUCTION

Presently the classic Hamiltonian quaternions of rotations of solid body (SB) with the parameters of Euler (Rodrigues–Hamilton) find application in the tasks of orientation of moving objects. They are rationed, have a single norm and can not be null [1].

## II. PROBLEM STATEMENT

Two types of the relatively new [2], [3], unrationed non-Hamiltonian quaternions of helf-rotations of SB are examined: $U = u_0 + \overline{\lambda}$, $V = v_0 + \overline{\lambda}$ where $u_0 = 1 - \lambda_0$, $v_0 = 1 + \lambda_0$; $\lambda_0 = \cos(\varphi/2)$, $\overline{\lambda} = \lambda \overline{k}$, $\lambda = \sin(\varphi/2)$; $\overline{k}$ is the unit vector of Euler axis of finite rotation (turn) of SB in three-dimensional vectorial Euclidean space; $\varphi$ is the angle of Euler finite rotation. Parameter $\lambda_0$ and coordinates $\lambda_n$ ($n$ = 1, 2, 3) of three-dimensional vector $\overline{\lambda}$ (in connected with SB coordinate orthonormal base) – are the material classic parameters of Euler (Rodrigues–Hamilton) as real numbers [1], [2]. They determine the quaternion of complete rotation $\Lambda = \lambda_0 + \overline{\lambda}$ with a unit norm $\|\Lambda\| = \lambda_0^2 + \lambda^2 = 1$, , $\lambda^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$ .

Quaternions $U$, $V$ emerge as a result of multiplication of the unconventional (new) rationed quaternions of helf-rotations $P = m + \overline{p}$ , $M = p + \overline{m}$ ( $m = \sin(\varphi/4)$, $\overline{p} = (\cos(\varphi/4))\overline{\kappa}$, $p = \cos(\varphi/4)$, $\overline{m} = (\sin(\varphi/4))\overline{\kappa}$ ) correspond-dingly by modules $|U| = 2m$ , $|V| = 2p$ , ($U = |U|P, V = |V|M$ ). At that the rationed quaternions $P$, $M$ are considered as material unit vectors in four-dimensional vectorial Euclidean space.

In contrast to quaternions $\Lambda$ unrationed quaternions $U$, $V$ may be null (if $\varphi = 0$ and $\varphi = 2\pi$ correspondingly), their modules and norms depend on the angle $\varphi$ . They represent practical interest for the solving two main problems: determination of SB orientation and control of SB orientation under condition of providing the shortest turns of SB (for angles $\varphi < 0$ and $\varphi > \pi$ ) in strapdown inertial orientation systems (SIOS) and control of orientation of moving objects [11].

## III. SOLUTION OF THE PROBLEM

### *A. Application of nonclassical quaternions for determination of orientation*

To determine the orientation the computer computational algorithms SIOS are used [1]. One-step algorithms of the third and fourth orders of exactness with the “scaled” quaternion of the kind $0,5U$ have been used, for example, in a scientific and production association “Khartron” (Kharkov, Ukraine) in the task of determination of orientation of space vehicle [3].

Particular practical interest is presented by the four-step algorithms of the fourth – sixth orders of exactness [1], providing possibility of recurrent calculations of quaternions $U$, $V$ with step $H = 4h$ ($h$ – a permanent and minimum possible step of discretization the signals of integrating gyroscopes by time in the computer of SIOS). The article [5] shows that the four-step algorithms are more efficient when used in SIOS than the one-step, two-step and three-step algorithms. The intermediate parameters of orientation are used in these algorithms [1] – coordinates $\varphi_{N+4,n}$ ($n = 1, 2, 3$) of a smaller vector $\overline{\varphi}_{N+4}$, characterizing the *Euler finite rotation* of the object to smaller angle in a time equal to step $H$. The algorithms of calculations of these parameters can be presented by one generalized four-step algorithm as

$$\varphi_{N+4} = q_{N+4} + a_1 Q_{-1} q_1 + a_2 Q_{-2} q_2 + a_3 \left(Q_{-2} q_1 + Q_{-1} q_2\right) + a_4 \left(Q_{-2} q_{-1} + Q_1 q_2\right), \quad (1)$$

where $q_{N+4} = q_{-2} + q_{-1} + q_1 + q_2$, $q_{-2}, q_{-1}, q_1, q_2$ are column matrices (1×3), made from the increases of corresponding angular quasicoordinates-signals of the gyroscopes formed in a side computer SIOS or SINS on four sequenced "smaller steps" $h$ of the questioning of gyroscopes; $Q_{-2}, Q_{-1}, Q_1, Q_2$ – corresponding skewsymmetric matrices. Values of permanent coefficients $a_v$ ($v$ = 1–4) in the algorithm (1), determining the concrete form of the examined examples of the algorithms of the fourth order of exactness, represented in the Table I. The algorithms 1, 2, 3, 5 are described in the monography [1], algorithm 4 – in the article [6] ("smoothing" algorithm of the fourth order, got on the basis of Tchebyshev polynomials). Algorithm 6 is a new four-step algorithm (optimal conical on exactness) [12].

TABLE I

THE CONSTANT COEFFICIENTS OF FOUR-STEP ALGORITHMS

| Factors | Number of algorithm | | | | | |
|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 |
| $a_1$ | 0 | 0 | 22/45 | 184/315 | –74/45 | 534/945 |
| $a_2$ | 16/9 | 0 | 22/45 | 112/315 | –9/2 | 486/945 |
| $a_3$ | 0 | 4/3 | 22/45 | 212/315 | 86/45 | 414/945 |
| $a_4$ | 0 | 0 | 32/45 | 52/105 | 0 | 696/945 |

Next equations is an example of the recurrent four-step algorithm 6-th order [1]:

$$\lambda_{N+4} = \frac{1}{2}\left(1 - \frac{1}{24}\varphi_{N+4}^2 + \frac{1}{1920}\varphi_{N+4}^4 - \frac{1}{322560}\varphi_{N+4}^6\right)\varphi_{N+4}, \quad (2)$$

$$\varphi_{N+4} = q_{N+4} + \left(\frac{22}{45} + \frac{1}{90}q_{N+4}^2\right)(Q_{-2} + Q_{-1})(q_1 + q_2) + \frac{32}{45}\left(Q_{-2}(q_{-1} + Q_{-1}q_2) - Q_2(q_1 + Q_{-2}q_1)\right), \quad (3)$$

$$\delta\hat{\lambda}_{N+4} = \frac{1}{7560}\sum_{k=2}^{6}\delta\lambda_{N+4}^{k}, \quad (4)$$

$$\delta\lambda_{N+4}^{(2)} = 32x_{05} + 192x_{11} + 256x_{23}, \quad (5)$$

$$\delta\lambda_{N+4}^{(3)} = 192x_{004} + 768x_{103} - 2112x_{112} - 3712x_{202} - 960x_{301} + 1344x_{013}; \quad (6)$$

$$\delta\lambda_{N+4}^{(4)} = 256x_{0003} + 4096x_{0102} - 1024x_{0201} + 9216x_{1101} - 9216x_{1002}, \quad (7)$$

$$\delta\lambda_{N+4}^{(5)} = -2048x_{00002} + 8192x_{100001} - 2048x_{01001}, \quad (8)$$

$$\delta\lambda_{N+4}^{(6)} = 2048x_{000001}, \quad (9)$$

where $x_{i_m \ldots i_2 i_1} = h^{\rho}\Omega_n^{(i_m)} \ldots \Omega_n^{(i_2)}\omega_n^{i_1}$; $\rho$ is the degree step $h$; $i_1 \neq i_2$; $i_m = 0, 1, 2, \ldots$; $i_m + \ldots + i_2 + i_1 + m = \rho$; $\Omega_n^{(j)} \simeq \omega_n^{(j)}$; $\omega_n^{(j)}$ is the time derivatives of the matrix $\omega_J(t_n)$, relating to the time point $t_n$; $\omega_n^{(0)} = \omega_n = \omega(t_n)$; $\omega_J(t_n)$ is the column matrix consisting of the coordinates of the absolute angular velocity of the object $\bar{\omega}(t)$ in a certain basis $\boldsymbol{J}$.

In Table II the values of constant of speed of calculable drift of algorithms are showed for comparison (at the conical vibrations of block of gyroscopes SIOS with conditions [6]: corner of nutation – 1 deg., frequency of conical vibrations – 10 Hz, step of calculations – 0,01 s), got within the hundredth stakes of percent at a computer design by the method of parallel account [1, p. 218].

TABLE II

THE CONSTANT VELOCITY OF THE DRIFT COMPUTING OF FOUR-STEP ALGORITHMS

| Option | Number of algorithm | | | | | |
|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 |
| The actual order of accuracy | 4 | 4 | 6 | 6 | 6 | 6 |
| The drift velocity, deg/h | 2.5 | 1.4 | $3.9\cdot10^{-4}$ | $9.6\cdot10^{-2}$ | $1.1\cdot10^{-2}$ | $2.2\cdot10^{-5}$ |

As seen in Table II, algorithm 3 substantially excels at exactness other algorithms (except for new algorithm 6) and is, essentially, a conical [7] algorithm (actually of the sixth order of exactness). An additional analysis showed advantages of algorithm 3 also in speed of operation. Optimal conical algorithm 6 exceeds in exactness even the four-step algorithm Litton [7], and his calculable complication is equal to calculable complication of algorithm 4.

Algorithm 3 (as main part of algorithm of calculations of Rodrigues–Hamilton parameters) is realized [8, p. 316] in the laser aviation strapdown inertial navigation system SINS-85, serially [9] produced since 2002 and intended for the use on the airplanes Il-96-300, Tu-204, Tu-334. Modifications of the system SINS-85 (SINS-77, SINS-T, SINS SP-1, SINS SP-2) are used on the airplanes An-70, Tu-95, Tu-160, Tu-214, Su-35, T-50, Yak-130 [10].

*B. Application of nonclassical quaternions for orientation control*

Parameters of quaternions *U, V* can be effectively used for the solving the problems of the orientation control of space vehicle (SV), as a solid body, in the positive definite quaternion functions of Lyapunov $f_u$ and $f_v$ of quadratic kind [2]:

$$f_u = \alpha_u u_0^2 + \beta_u(\bar{\lambda}\cdot A_u\bar{\lambda}) + \gamma_u(\bar{\omega}\cdot\bar{g}),$$
$$f_v = \alpha_v v_0^2 + \beta_v(\bar{\lambda}\cdot A_v\bar{\lambda}) + \gamma_v(\bar{\omega}\cdot\bar{g}), \quad (11)$$

where $\alpha_u$, $\beta_u$, $\gamma_u > 0$ and $\alpha_v$, $\beta_v$, $\gamma_v > 0$; $A_u$, $A_v$ are positive definite symmetric permanent operators; $g = J\bar{\omega}$ is the vector of kinetic moment SV; *J* is the operator (tensor) of inertia SV; $\bar{\omega}$ is the angular velocity vector of SV.

To provide the control of the shortest turns of SV the function $f_u$ at $u_0 < 1$, $v_0 > 1$ $(0 < \varphi < \pi)$ or the function $f_v$ at $u_0 > 1$, $v_0 < 1$ $(\pi < \varphi < 2\pi)$ is used.

*C. Pentanions of helf-rotations*

On the basis of rationed Λ and unrationed U, V quaternions there turn out the new five-dimensional vectors of helf-rotations of a kind $\bar{x} = x_0\bar{i}_0 + \bar{\lambda} + x_4\bar{i}_4$, where $\bar{\lambda}$ is three-dimensional vector $\bar{\lambda} = \lambda_1\bar{i}_1 + \lambda_2\bar{i}_2 + \lambda_3\bar{i}_3$ in the quaternions *Λ, U, V*; $x_0$, $x_4$ – two any scalar parameters out of three: $u_0$, $v_0$, $\lambda_0$; $\bar{i}_1 \ldots \bar{i}_4$ are unit vectors of some five-dimensional orthonormal coordinate base; $x_0, \lambda_1, \lambda_2, \lambda_3, x_4$ are coordinates $x_m$ ($m$ = 0, 1, 2, 3, 4) of vector $\bar{x}$ in this base.

To solve the problems of SB orientation control, for example, the five-dimensional vector can be used $\bar{x}_{UV} = u_0\bar{i}_0 + \bar{\lambda} + v_0\bar{i}_4$ with the formulas of multiplying:

$$u_0 = u_0' + u'' - (u_0'u_0'' - (\bar{\lambda}'\cdot\bar{\lambda}'')),$$
$$\bar{\lambda}_0'' = 0,5(v_0'' - u_0'')\bar{\lambda}' + 0,5(v_0' - u_0')\bar{\lambda}'' + (\bar{\lambda}'\cdot\bar{\lambda}''), \quad (12)$$
$$v_0 = 2 - v_0' - v_0'' + v_0'v_0'' - (\bar{\lambda}'\cdot\bar{\lambda}'').$$

Here $0,5(v_0'' - u_0'') = \bar{\lambda}_0''$, $0,5(v_0' - u_0') = \bar{\lambda}_0'$, $(\bar{\lambda}'\cdot\bar{\lambda}'')$ are scalar product of vectors $\bar{\lambda}'$ and $\bar{\lambda}''$ of the first and the second rotation.

The systems of kinematic differential equations for five-dimensional vectors are linear and one of them looks like, for example, in a scalar-vectorial record: $2\dot{u}_0 = (\bar{\omega}\cdot\bar{\lambda})$, $2\dot{\bar{\lambda}} = 0,5(v_0 - u_0)\bar{\omega} + \bar{\lambda}\times\bar{\omega}$; $2\dot{v}_0 = -(\bar{\omega}\cdot\bar{\lambda})$.

Out of five-dimensional vectors $\bar{x}$ of helf-rotations (by analogy with quaternions) the hypercomplex [8] five-dimensional systems – pentanions of SB helf-rotations appear [11]. Any pentanion is written down as a hypercomplex number (without unit vector $\bar{i}_o$) as $X = x_0 + \lambda_1\bar{i}_1 + \lambda_2\bar{i}_2 + \lambda_3\bar{i}_3 + x_4\bar{i}_4 = x_0 + \bar{\lambda} + x_4\bar{i}_4$, where $x_0$ is the scalar part of pentanion, $(\bar{\lambda} + x_4\bar{i}_4)$ is the vector part, $x_m$ are pentanion parameters. The norm $\|X\|$ of pentanion is determined by the scalar product:

$$\|X\| = (\bar{x}\cdot\bar{x}) = x_0^2 + (\bar{\lambda}\cdot\bar{\lambda}) + x_4^2.$$

*Pentanions of helf-rotations* have a row of advantages in contrast to classic five–dimensional parameters of Khopf orientation [11], got from six direction cosines of SB unit vectors.

## IV. Conclusions

Thus, the possibility of application of parameters of nonclassical quaternions and pentanions of SB helf-rotations is shown in the tasks of control and orientation of moving objects. Unlike the classic rationed quaternions of rotations the considered nonclassical quaternions of semirotations can be null and their modules and norms depend on the corner of Euler finite rotation of SB. Due to the special properties the nonclassical quaternions and pentanions of helf-rotations can be effectively used in the algorithms of the strapdown inertial systems of orientation and orientation control systems along with the classic rationed Hamilton quaternions of rotations or instead of them.

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**А. П. Панов, С. О. Пономаренко. Нетрадиційні кватерніони і пентаніони в задачах інерціальної орієнтації**

Розглянуто некласичні кватерніони і пентаніони напівобертання твердого тіла та їх застосування в задачах керування і орієнтації рухомих об'єктів. На відміну від класичних нормованих гамільтонових кватерніонів повних обертань некласичні кватерніони напівобертання можуть бути нульовими, вони мають змінні норми, що залежать від кута ейлерового кінцевого обертання.

**Ключові слова:** некласичні кватерніони; пентаніони напівобертання; безплатформенні інерціальні системи орієнтації; керування орієнтацією.

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**А. П. Панов, С. А. Пономаренко. Нетрадиционные кватернионы и пентанионы в задачах инерциальной ориентации**

Рассматрены неклассические кватернионы и пентанионы полувращений твердого тела и их применение в задачах управления и ориентации движущихся объектов. В отличие от классических нормированных гамильтоновых кватернионов полных вращений неклассические кватернионы полувращений могут быть нулевыми, они имеют переменные нормы, зависящие от угла эйлерова конечного вращения.

**Ключевые слова:** неклассические кватернионы; пентанионы полувращений; бесплатформенные инерциальные системы ориентации; управления ориентацией.

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