The investigation process of the geological object by mathematical modeling and computational experiment is approximate. There are certain errors on every research stage. The development of effective methods of calculations and high qualification of researchers-calculators improves the final result accuracy.

Apriori it can be argued that the results containing the smallest deviation of calculated values from the true values are more suitable for researchers. On this basis the requirements to the accuracy of calculations are sometimes too high and do not correspond actual necessity and real opportunities.

Absolutely faultless results can not be obtained. Therefore, the computational process must be making such way as that the results were obtained with a given accuracy. The notion of “given accuracy” should be accompanied by a certain numerical criteria.

Key questions in the theory of error’s calculations can be formulated as follows:

– study of the laws of occurrence and distribution of computational errors;
– estimation of expected accuracy and precision definition of the final results of calculations.

The choice of physical and mathematical models is inevitably connected with the simplification of initial physical phenomenon, insufficiently precise specification of the equation coefficients and other initial data. Relatively the numerical method by which it is investigated this mathematical model these errors are irremovable. It is impossible to avoid them in this model [1].

At the transition from a mathematical model to the numerical method there is the method error [1]:
– discretization error;
– rounding error.

At present, the finite element method (FEM) is widely used in decision various problems of mathematical physics. The main drawback of any variational method, in particular FEM, is the difficulty of obtaining initial estimates. The method reliability is verified by test of each program on the exact solutions [2].

The solution obtained on the basis of the finite element discretization inevitably differs from the exact. Sources of variation are [3]:
1) finite-element discretization of the space;
2) approximation of the basic functions that are defined elementwise by local functions;
3) rounding errors.

Value of errors of group 1 can be reduced if it thickens the finite element mesh in areas of high gradients.

Errors of group 2 are determined by the finite element type. If not change of the element type value of such errors is reduced by increasing number of partitions of space on the elements. Accuracy increased of solution approximation at fixed finite elements mesh can be reach by successive degree increase of basis functions.

Both the first group of errors in considerable degree depends on the researcher experience in constructing the finite-element mesh. The value of these errors can be decreased to an acceptable level in the problem formulation.
Errors of group 3 are purely mathematical. Finite-dimensional analogue of the initial mathematical problem – discrete model – is a system of large number of algebraic equations. The system solution exactly and in an explicit form it is impossible to find. The system input (the coefficients and right parts) are given into the computer with rounding. Rounding error or computational error in the work process of the algorithm has been accumulating. The computational error value is determined by two factors [1]:

- the definition accuracy of real numbers in the computer;
- the algorithm sensitivity to rounding errors.

Single-value answer to the question – which of the three errors (model error, method and computer) is the predominant – it is impossible to give.

In solving problems of mathematical physics has been arising following situations [1]:

- model error significantly exceeds the method error, then the rounding error in the case of stable algorithms can be neglected as compared with the method error;
- for solving systems of ordinary differential equations are used as exact methods, that their accuracy is comparable with the rounding error.

In general, it is necessary hold to following strategy: given errors should be the same order [1].

**Work purpose** – method development for more accurate definition of finite element solution based on practical analysis of the distribution of errors.

**Problem statement**

Classes of finite elements differ from each other in number of degrees of freedom. The question has been arising: is it possible to get the advantage of economic or other nature, complicating element through the increase number of degrees of freedom.

For a given accuracy extent complication of the element leads to a decrease general number of unknowns. However, algorithm efficiency is defined as the account time and the preparation complication extent of input data [2-4].

On the one hand, the elements must be chosen enough small to obtain acceptable results. On the other hand, the use of sufficiently large elements reduces the computational work.

It is necessary to have some general considerations about the final approximate values in order to be able to reduce the size of the elements in those areas where the expected result can strongly vary (high values of gradients), and increase them where the expected result is almost constant.

**Analysis of received results**

To illustrate the possible accuracy in practice as an example let’s consider a two-layer environment (fig. 1).

![Diagram](image)

Fig. 1. Two-dimensional problem of heat conduction in two-layer environment

Dimensions of the environment are given in absolute units. Thermal conductivity \( k \), heat generation \( Q \) and heat flow \( q \) close to the real values. It is required to find the temperature distribution in this area.

Defining equation for this problem is the Poisson equation

\[
\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + Q = 0
\]

with Dirichle boundary conditions on the upper part of the boundary

\[
T = 0^0\text{C}, Y = 0\text{km}
\]

and Neumann conditions on the rest part

\[
k \frac{\partial T}{\partial x} = 0, X = 0\text{km} ;
\]

\[
k \frac{\partial T}{\partial x} = 0, X = 60\text{km} ;
\]

\[
k \frac{\partial T}{\partial y} = 20\text{mW/m}^2, Y = 40\text{km}.
\]

In principle, one can to obtain an exact solution here, because conditions do not change in direction \( X \), and the problem can be viewed as a one-dimensional:
\[ k \frac{\partial^2 T}{\partial y^2} + Q = 0. \]

Definition domain of the problem is the interval \(0 \leq Y \leq 40 \text{ km}.\) Analytical solution of differential equations for the one-dimensional case has the form

\[ T = T_s + \frac{q_s}{k} y - \frac{Q}{2k} y^2, \]

where \(T_s, q_s\) – the temperature and heat flow on the model surface, respectively.

For this problem a finite element mesh contains 48 elements and 35 nodes (fig. 2).

Nodal parameters of triangular elements are the values of temperature in nodes. Such elements are called lagrangian.

Fig. 3 shows the results of solving equation (1) with boundary conditions (2)–(5).

Comparison this results with the exact solution (the values in the second line) shows that using the linear approximation of the temperature at the area division on 48 elements leads to a solution that differs from the analytical not more than 0.6%.

Heat conduction problem can be solved in two-layer environment also using elements of other polygonal shapes.

In many cases it is expedient to use as parameters not only the values of the function, but also the values of its derivatives. These are so-called hermitian elements [4].
Full cubic trial function can be written as:

\[
\hat{T}_e = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z + \alpha_5 x^2 + \alpha_6 y^2 + \alpha_7 z^2 + \alpha_8 xy + \alpha_9 yz + \alpha_{10} xz
\]

where the subscript \( L \) indicates the local coordinate system.

To determine the 10 constants \( \alpha_1, \alpha_2, \ldots, \alpha_{10} \), the element must have 10 nodal parameters. As the parameters have chosen three values of the function and its first derivatives in each node together with the function value in the mass center \( C \). With this choice trial functions are continuous and their first derivatives are piecewise continuous throughout the domain [4].

Node lying in the mass center affects only on the element contribution to which it belongs. From each elemental matrix equation one can to eliminate the nodal parameter, belong to the mass center. This process is called condensation.

Each of the functions \( \{T\}^e \) can be divided into two parts, one of which \( \{T'\}^e \) is connected with neighboring elements, and the other \( \{T^C\}^e \) is typical only for this element. Then one can to write

\[
\frac{\partial \Pi}{\partial \{T^C\}^e} = \frac{\partial \Pi^e}{\partial \{T^C\}^e} = 0 ,
\]

\[
\Pi^e = \frac{1}{2} T^T kT + T^T F^e ,
\]

where \( k \) – elemental stiffness matrix,

\( F^e \) – the contribution of the internal distributed load.

The procedure for exception \( \{T^C\}^e \) from the further consideration can be carried as follows.

\[
\frac{\partial \Pi^e}{\partial \{T\}^e} = \begin{bmatrix}
\frac{\partial \Pi^e}{\partial \{T'\}^e} \\
\frac{\partial \Pi^e}{\partial \{T^C\}^e}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial \Pi'}{\partial \{T'\}^e} \\
0
\end{bmatrix} = \begin{bmatrix}
k_{ij} \\ k_{i,10}
\end{bmatrix} \begin{bmatrix}
\{T'\}^e \\
\{T^C\}^e + \{F^C\}^e
\end{bmatrix} = \begin{bmatrix}
k_{i,j} \\ k_{i,10}
\end{bmatrix} \begin{bmatrix}
\{T'\}^e \\
\{T^C\}^e + \{F^C\}^e
\end{bmatrix} ,
\]

where \( i = 1, 2, \ldots, 10, j = 1, 2, \ldots, 9 \).

From the second row of (6) one can be finding

\[
T^C = -\frac{k_{i,10}}{k_{10,10}} T_j^' - F^C .
\]

Then

\[
\frac{\partial \Pi}{\partial T_i^e} = k_{ij} T_j^' - \frac{k_{i,10} k_{10,j}}{k_{10,10}} T_j^' + F_i - \frac{k_{i,10}}{k_{10,10}} F^C . \quad (7)
\]

Itself is excluded from the matrix equation explicitly and reliably hidden in the expression (7).

From equation (7) follows that the elements of condensed stiffness matrix \( k' \), denoted \( k' \), is given by

\[
k_{ij}' = k_{ij} - \frac{k_{i,10} k_{10,j}}{k_{10,10}}, i = 1, 2, \ldots, 9, j = 1, 2, \ldots, 9 .
\]

A distributed load vector element has the form

\[
F_i' = F_i - \frac{k_{i,10}}{k_{10,10}} F^C , i = 1, 2, \ldots, 9 .
\]

Fig. 3 shows the results of solving equation (1) with boundary conditions (2)–(5) with cubic approximation of the trial function. The obtained values are in the second row and are identical in all nodes with the analytic solution of the problem.

Let’s consider next example of vertical contact between two environments with thermal conductivity \( k_1 \) and \( k_2 \) having of analytical solution [5; 6]. The problem geometry is shown in fig. 5.

![Fig. 5. Vertical contact model of two environments](image_url)

Dimensions of the model are presented in relative units and are normalized by the ledge power \( L \). Heat flow is normalized by specified on the lower boundary the regional value \( q_0 = 50 \text{ mW/m}^2 \).

Defining equation for this problem is the stationary thermal conductivity equation of the following form:
The practical analysis has shown that, with the large problems in order to save an operational memory to improve the accuracy of calculations appropriate to apply a more accurate partition of mesh with the routine use of triangular elements.

The approximation choice depends on the specific task. It will be the best approximation which gives the most accurate solutions with the least computational costs.

References


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