SYNTHESIS OF ROBUST CONTROLLER FOR STABILIZING SYSTEM OF INFORMATIONAL-MEASURING DEVICES

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Abstract. The paper is devoted to the actual issue of controller synthesis for the stabilizing systems of the information-measuring devices meant for exploitation at vehicles of a wide class. In the paper the problem statement has been represented. Features of the $H_\infty$-synthesis procedure for the studied class system based on the method of mixed sensitivity have been defined. Efficiency of the suggested approach has been proved by the simulation results. The obtained results may be applied in the field of the wide class information-measuring devices stabilizing systems.

Keywords: informational-measuring device, method of mixed sensitivity, robust controller, stabilizing system.

Introduction

Nowadays, the complexity of control processes attended exploitation of vehicles sufficiently increases. At that the important problem of stabilization of the informational-measuring devices providing measurements and obtaining of information needed for control by vehicles, navigation, tracking by the references points arises. Usually the higher rigid requirements are given to such processes. It is impossible to satisfy these requirements without stabilization of base on which the appropriate informational-measuring devices are mounted.

It should be noted, that accuracy characteristics of the informational-measuring devices are sharply improved during the last years. Such tendency requires appropriate progress in solving the problem of their stabilization during exploitation at vehicles. Moreover, one of the most important problems is synthesis of the controller for stabilizing system of the informational-measuring devices.

Nowadays, methods of control systems synthesis, based on the advanced engineering technique, which uses models in the state space and the complex mathematical procedures of optimal controller determination, are widespread. This approach requires the extremely complex mathematical apparatus. But taking into consideration availability of the appropriate software which implements this apparatus, creation of controllers with the complex structure is considerably simplified for a developer. Nowadays, there are many methods and approaches to creation of the wide class controller on the base of the advanced engineering technique.

The comparative analysis of the different approaches to synthesis of the wide class control systems is represented in table.

Notice, that all represented methods ensure definition of the explicit control laws and have powerful dataware and software in the form of such MATLAB’s packages as Control System Toolbox, Robust Control Toolbox, $\mu$-Analysis and Synthesis Toolbox.

Analysis of the last researches

Approaches to design of controllers for the information-measuring devices stabilizing system are represented in many works. Problems of the robust controller synthesis are represented in [1; 2]. Statement of the canonical robust control problem and representation of the typical uncertainties is given in [3].

Characteristic of the Control System Toolbox and the Robust Control Toolbox with corresponding examples is given in [4].

Characteristic of the method of mixed sensitivity and approach to the weighting transfer functions choice are given in [5].

Approaches to robust controllers synthesis are represented in [6; 7].

The goal of this paper is a choice of the method of synthesis of the robust controller for the stabilizing system of the wide class information-measuring devices, destined for exploitation at vehicles, and determination of basic features of this method implementation.
Comparative analysis of the optimal synthesis methods

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Synthesis of the robust controller

The choice of the synthesis method essentially depends on features of the system and conditions of its exploitation. Firstly, the studied systems operate in the conditions of the external disturbances actions (irregular sea, influence of wind and disturbances, caused by irregularities of the road profile, for ships, aircrafts and ground vehicles respectively). Secondly, parameters of these control objects significantly change during exploitation.

Taking into account these features, the problem of controller synthesis for the studied system may be solved on the base of the robust control. The main task of the system’s robust control synthesis is search of the control (stabilization) law, which is capable to provide a system’s accuracy performances in the given limits in spite of uncertainties in a system’s mathematical description. This uncertainty can be caused by different factors such as: the external disturbances, errors of a system’s transfer function determination and non-modeled dynamics.

Effective procedures of robust controller creation can be obtained on the base of $H_\infty$ - synthesis.

The $H_\infty$-control is one of the most widespread advanced engineering techniques. It was introduced by Zames [8]. Nowadays this technique is used in order to achieve robust performance. The control problem is represented as a mathematical optimization problem.

Some approach to the robust systems creation exists, which is based on a system’s transfer function singular values determination and minimization of the corresponding norm [6]. To make this norm definite all corresponding transfer functions must be proper. This approach to robust control systems design in general and to stabilizing system design in particular can be realized by automated means of optimal design such as Robust Control Toolbox of the system MATLAB.

The canonical robust control problem statement is illustrated by the fig. 1 [3].

Fig. 1. Canonical robust control problem statement:
- $\Delta$ – uncertainty;
- $P_{nom}$ – the transfer function of the nominal control object;
- $F$ – the transfer function of the controller;
- $u$ – the vector of controls;
- $y$ – the vector of outputs
For the disturbed system represented in fig. 1 the state and output equations may be represented in the following form:

\[
\begin{align*}
\dot{x} &= Ax + B_1u_1 + B_2u_2; \\
y_1 &= C_1x + D_{11}u_1 + D_{12}u_2; \\
y_2 &= C_2x + D_{21}u_1 + D_{22}u_2
\end{align*}
\]

or in the matrix form

\[
\begin{bmatrix}
\dot{x} \\
y_1 \\
y_2
\end{bmatrix} =
\begin{bmatrix}
A & B_1 & B_2 \\
C_1 & D_{11} & D_{12} \\
C_2 & D_{21} & D_{22}
\end{bmatrix}
\begin{bmatrix}
x \\
u_1 \\
u_2
\end{bmatrix}.
\]

The statement of the robust control problem may be generalized by integration of the nominal object and the uncertainty as it is shown in fig. 2.

![Fig. 2. Statement of the generalized robust control problem](image)

The matrix of the augmented control object becomes

\[
\begin{bmatrix}
A & B_1 & B_2 \\
C_1 & D_{11} & D_{12} \\
C_2 & D_{21} & D_{22}
\end{bmatrix} =
\begin{bmatrix}
P_{11} & P_{12} \\
P_{21} & P_{22}
\end{bmatrix}
\]

According to the fig. 2 the problem of \( H_\infty \)-controllers design may be formulated in such a way [7]. For the given augmented object \( P(s) \) with the mathematical description in the state space it is necessary to determine the stabilizing controller in the feedback loop for the control object

\[
u_2(s) = F(s)y_2(s),
\]

that minimizes \( H_\infty \)-norm of the closed matrix transfer function

\[
\begin{align*}
T_{yu1}(s) &= P_{11}(s) + P_{12}(s)(I - F(s)P_{22}(s))^{-1} \\
&\quad \times F(s)P_{21}(s),
\end{align*}
\]

that is

\[
\| T_{yu1} \|_\infty = \sup_{\omega} \sigma_{\max} (T_{yu1}(j\omega)) < 1,
\]

where \( \sigma_{\max} \) – the maximal singular value.

\( H_\infty \)-synthesis is the powerful tool for design of control systems with feedback on the base of shaping of the frequency responses as functions of the singular values. There is an approach to the robust systems design, when the sufficient condition of the robust stability is formulated in the form of the norm, bounded by the weighting transfer functions. This approach is accepted in the Robust Control Toolbox [1], which is the computer-aided facility of optimal robust systems design.

In most cases process of robust systems design may be estimated only by the upper limit of the transfer function or frequency response deviation from nominal one. Different approaches to determination of the bounded weight frequency responses are known. One of the methods for obtaining of such bounds is using the results of the experimental researches on the base of which the real frequency response is determined. But in most cases information about the frequency responses of a real system is not available. More often the approach based on the frequency requirements to a system is used for determination of the bounded responses.

The singular values of the closed matrix transfer functions from the control signal \( r \) to the signals of an error and input and output signals of \( e, u, y \) can be used for the numerical estimation of the stability margins and frequency responses of a system. These transfer functions may be described by the following expressions [7]

\[
\begin{align*}
S(j\omega) &= (I + F(j\omega)P(j\omega))^{-1}, \\
R(j\omega) &= F(j\omega)(I + F(j\omega)P(j\omega))^{-1}, \\
T(j\omega) &= F(j\omega)P(j\omega)(I + F(j\omega)P(j\omega))^{-1}.
\end{align*}
\]

The matrices \( S(j\omega) \) and \( T(j\omega) \) are called the sensitivity function and the complementary sensitivity function correspondingly.
The singular values of the sensitivity function define the level of a disturbances attenuation because it represents the transfer function of the closed system from the disturbance \( w \) to the output signal \( y \). The level of the disturbances attenuation may be estimated by the formula
\[
\sigma(S(j\omega)) \leq |W_1^{-1}(j\omega)|,
\]
where \( |W_1^{-1}(j\omega)| \) is the desirable level of disturbance attenuation.

The singular values of the functions \( R(j\omega) \) and \( T(j\omega) \) can be used for estimation of the stability margins of the system under action of additive and multiplicative uncertainties.

As a rule, influence of all uncertainties on the object is estimated by the single multiplicative uncertainty \( \Delta_M \). Then the requirements to control system design can be defined in the following way [7]:
\[
\frac{1}{\sigma_i(S(j\omega))} \geq |W_1(j\omega)|,
\]
\[
\sigma_i(T(j\omega)) \leq |W_3^{-1}(j\omega)|.
\]

At that the following condition must be satisfied:
\[
\sigma(W_1^{-1}(j\omega)) + \sigma(W_3^{-1}(j\omega)) > 1 \quad \text{for any } \forall \omega.
\]

After weighting matrix transfer functions choice the studied system may be augmented by these functions. The augmented transfer function of the system may be described by the scheme, represented in fig. 3.

![Diagram](image)

Fig. 3. The augmented transfer function of the system:
\( z_1, z_2, z_3 \) – additional outputs

For the method of mixed sensitivity the requirements to the disturbance attenuation and provision of the stability margin are reduced to the unique requirement [7]:
\[
\|T_{y\mu_1}\|_\infty \leq 1,
\]
where
\[
T_{y\mu_1} \overset{\text{def}}{=} \begin{bmatrix} W_1S \\ W_2R \\ W_3T \end{bmatrix}
\]
is the cost function of the method of mixed sensitivity. This function defines penalties for both the sensitivity function and the complementary sensitivity function.

Synthesis of the robust controller may be considered by example of the system for stabilization of the informational-measuring devices mounted at the ground vehicles. Such systems operate under action of disturbances. The design of the robust controllers is an actual problem for this application.

The joint model of the plant for such system includes the actuating mechanism, stabilization object and the measuring system. This model can be represented as the set of the following differential equations:
\[
J_e \ddot{\varphi}_e = -M_f + \frac{c_m}{R_{arm}} U + \frac{c_n}{n_t} \varphi_e - c_t \varphi_e;
\]
\[
J_{imd} \ddot{\varphi}_{imd} = -M_f + \frac{c_m}{n_t} \varphi_{imd} - c_t \varphi_{imd} - M_{un};
\]
\[
\dot{U}_{arm} + U = -c_e \dot{\varphi}_e + U_{PWM};
\]
\[
\dot{U}_g T_g^2 + \dot{U}_g 2\zeta T_g + U_g = \varphi_{imd},
\]
where \( J_e \) – the moment of inertia of the engine;
\( \varphi_e \) – the angle of engine rotation;
\( M_f \) – the nominal engine antitorque moment;
\( c_m \) – the constant of the loading moment at the engine shaft;
\( R_{arm} \) – the resistance of the engine armature winding;
\( U \) – the voltage of the engine armature winding;
\( J_{imd} \) – the moment of inertia of the informational-measuring device;
\( \varphi_{imd} \) – the rotation angle of the informational-measuring device;
\( M_{un} \) – the unbalanced moment;
\( c_t \) – the reducer rigidity;
\( n_r \) – the ratio of the reducer; 
\( U_{PWM} \) – the voltage of the wide pulse modulator; 
\( c_e \) – the electromotive force constant.

Using the above stated advanced technique requires to linearize this model and represent it in the state space.

After linearization the moments \( M_r, M_f \) the model becomes:

\[
J_e \dot{\phi}_e = -f_e \phi_e + \frac{c_m}{R_{arm}} U + \frac{c_e}{n_r^2} \phi_e - c_r \phi_e;
\]

\[
J_{imd} \dot{\phi}_{imd} = -f_{imd} \phi_{imd} + c_e \phi_{imd} - c_r \phi_{imd} - M_{un};
\]

\[
\dot{U}_{arm} + U = -c_e \phi_e + U_{PWM};
\]

\[
\ddot{U}_g T_g^2 + \dot{U}_g 2 \zeta T_g + U = \phi_{imd},
\]

where \( f_{imd}, f_e \) are the coefficients of the linearized friction \( M_r = \text{sign} \phi_e \) and antitorque moments \( M_f = \text{sign} \phi_{imd} \) respectively.

The represented model can be transformed to the general form of the state space model

\[
\dot{x} = Ax + Bu,
\]

\[
y = Cx + Du,
\]

where \( x \) – the state vector;  
\( u \) – the vector of controls;  
\( y \) – the vector of the observations;  
\( A, B, C, D \) – the matrices which characterize features of the system, controls, observations and disturbances.

After introducing of new variables and reduction the order of the differential equations the state, control, observation and disturbance vectors and corresponding matrices look like:

\[
x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} \Phi_e \\ \Phi_{imd} \\ \phi_e \\ \phi_{imd} \\ U \\ \dot{U}_g \\ U_g \end{bmatrix},
\]

\[
u = \begin{bmatrix} M_{un} \\ U_{PWM} \end{bmatrix};
\]

\[
y = \begin{bmatrix} \phi_{imd} \\ U \end{bmatrix};
\]

\[
B^T = \begin{bmatrix} -1 \\ J_{imd} \\ T_{arm} \end{bmatrix};
\]

\[
C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \end{bmatrix};
\]

\[
D = \begin{bmatrix} 0 & 0 \\ \end{bmatrix};
\]

\[
A = \begin{bmatrix} -f_e & \frac{c_r}{n_r^2} & -c_e & -c_r & 0 & 0 & 0 \\ 0 & -f_{imd} & c_r & -c_r & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \end{bmatrix}.
\]

Results of simulation by the method of mixed sensitivity are represented in fig. 4.

The choice of the weighting matrix transfer functions is the ambiguous problem. Solution of this problem requires using of the heuristic methods, for example, the method of trials and errors, which takes into account experience of a system’s developer.

Accordingly to the scheme represented in the fig. 5 to damp the disturbances it is desirable to have small error \( e \) in the low frequency band and to damp the noise it’s desirable to have the small value \( y \) in the high frequency band. That is why for damping the error \( e \) in the low frequency band the amplitude of the weighting transfer function \( W_1 \) should decrease with increasing of the frequency. And the amplitude of the weighting function \( W_3 \) should increase with increasing of the frequency.
The weighting transfer function $W_2$ is usually used in order to limit control and regulate the operating speed. In some cases the inputting of $W_2$ is necessary for the problem solution [4]. It is possible to use function $W_2 = \varepsilon I$, where $\varepsilon$ is a small value, $I$ is the unitary matrix.

For the system to be studied the following weighting coefficients were used

$$W_1 = \frac{1}{s^2};$$
$$W_2 = 0.04;$$
$$W_3 = 0.1 \frac{0.1s + 1}{0.01s + 1}.$$

In the Robust Control Toolbox for finding $H_\infty$-norm two methods may be applied. In both cases for finding minimum of $T_{y\mu}$-norm the method of two Riccati equations is used for finding solutions $P$ and $S$ [1]. Such approach allows to find a controller which provides stability and the minimal sensitivity to disturbances of a plant with changing parameters.

The state and control estimation is carried out in accordance with the equations

$$\frac{dx_e}{dt} = (A + \mu^{-2}B_1B_2^TP)x_e + B_2u_2 + \mu L[y_2 - (C_2 + D_0)x_e];$$
$$u_2 = Kx_e.$$
where \( K, L, D_0 \) depend on the Riccati equation solutions \( P, S \) and \( D_{11} = D_{22} = 0 \).

It is necessary to mark, that for solution obtaining is necessary to introduce the weighting transfer function \( W_2 \) and give the quadratic matrix \( D_{21} \).

At the first approach the stabilizing controller that satisfies the condition (1) is determined and also is proved that the condition

\[
\| T_{y\mu 1} \|_\infty < \mu
\]

is true if \( x_\infty, y_\infty \) – nonnegative definite and

\[
\rho (PS) < \mu^2,
\]

where \( \rho \) – the spectral radius of the matrix product.

In the Robust Control Toolbox this approach is implemented by means of the instruction \( \text{hinf} \).

At the second approach in order to define the optimal controller the so-called \( \gamma \)-iterations are used

\[
\| \gamma_i T_{y\mu 1} \|_\infty < 1,
\]

where \( i \) – number of iteration.

In the Robust Control Toolbox this approach is implemented by means of the instruction \( \text{hinfopt} \).

This instruction implements the so-called \( \gamma \)-iterations, which allows to define the optimal controller based the relationships, used in the instruction \( \text{hinf} \).

Creation of systems for the informational-measuring devices stabilization is not a new problem. It may be solved using different approaches. But taking into consideration influence of the external and internal disturbance exploitation, the \( H_\infty \)-synthesis may be preferred for such problem solving. This method is one of the most advanced techniques for robust controllers design.

The basic disadvantage of this method is the necessity to use the weighting transfer functions. It worth noting that successful solving of the problem depends on the weighting transfer functions choice.

**Conclusions**

The \( H_\infty \)-synthesis problem statement for the stabilizing system of the wide class informational-measuring devices meant for exploitation at vehicles was defined. The weighting transfer functions were determined and the procedure of epy robust stabilizing system synthesis by the method of mixed sensitivity was carried out.

**References**


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