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APPROXIMATION OF BESSEL FUNCTIONS BY RATIONAL FUNCTIONS

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Abstract—*A problem of reducing expressions, which are received while solving the system of Navies - Stokes equations is discussed. An algorithm of equivalent reducing is proposed, based on the theory of chain fractions.*

Index Terms—Integral transforms; iterative procedure; convective-diffusion transform; Navies–Stokes equations; chain fractions; Bessel's functions.

I. INTRODUCTION

In the study of convection-diffusion processes, for example in the form of a cylindrical cyclone devices [1], there is a need for solving boundary value problems described by systems of nonlinear differential equations in partial derivatives that describe the movement of swirling flow in axisymmetric channels with proper initial and boundary conditions on the inner and outer walls of the channel. In this system of equations includes equations for the components of velocity, temperature and concentration of impurities contained in the test flow.

II. THE ANALYSIS OF THE LATEST RESEARCHES AND PUBLICATIONS

The traditional approach to solving such systems is the use of difference schemes [2]. Thus, first, ignoring the convective component, included in the equations describing the dynamics of flow, and secondly, the application of difference schemes for solving systems of nonlinear equations, leading to the solution of nonlinear functional equations even in the one-dimensional case, for example, with respect to the radial component, it gives satisfactory results only individual (model instances) as to achieve satisfactory accuracy is necessary to solve the cumbersome system of nonlinear algebraic equations.

III. TASK STATEMENT

Consider the following partial differential equation (one equation system describing the dynamics of flow in cylindrical channels).

$$\frac{\partial u_r}{\partial t}+u_r\left(\frac{\partial u_r}{\partial r}+\frac{1}{r}u_r\right)+u_\varphi\frac{1}{r}\frac{\partial u_r}{\partial\varphi}+u_z\frac{\partial u_r}{\partial z}$$
$$=\frac{\mu_{ef}}{\rho}\left[\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial\varphi^2}+\frac{\partial^2 u_r}{\partial z^2}\right]+\frac{1}{\rho}S_{u_r}, \tag{1}$$

$$S_{u_r}=\frac{\rho u_\varphi^2}{r}-\frac{\partial p}{\partial r}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\mu_{ef}\frac{\partial u_r^2}{\partial r}\right)+\frac{1}{r}\frac{\partial}{\partial\varphi}\left(\mu_{ef}\frac{r\partial(u_\varphi/r)}{\partial r}\right)-2\frac{\mu_{ef}}{r^2}\frac{\partial u_\varphi}{\partial\varphi}+\frac{\partial}{\partial z}\left(\mu_{ef}\frac{\partial u_z}{\partial r}\right), \tag{2}$$

Author of the work [4] proposed a numerical-analytical method for solving nonlinear partial differential equations, which is free from these weaknesses and gives solution in the form of a number (for the equation (1), taking into account the turbulent component (2)). The essence of the iterative procedure is to find solutions of the linear part of the equation by using integral transformations of spatial variables and time with future planned use of solutions to find a non-linear part. This raises the problem of computing the integral transforms of the product of two or more of their self-functions of the linear part of the problem, in particular, in the calculation of integrals of the product of cylindrical functions (Bessel functions of the first and second kind, and combinations thereof).

$$u^{(1)}(r,\varphi,\zeta,t)=\sum_{m_u,l_u}V_u(\beta_u,r)\Phi_u(\varphi)Z_u\left(\alpha^u_{m_u},\zeta\right)\left[U1^0_{m_u,l_u}+e^{\sigma^u_{m_u,l_u}t}\left[U1^1_{m_u,l_u}\varphi1^u_{m_u,l_u}(t)+U2^2_{m_u,l_u}\varphi2^u_{m_u,l_u}(t)\right]\right]. \tag{3}$$

While receiving this expression are evaluated

$$b_{i,j,k}=\int_{r_0}^{R}V_i(\beta,r)V_j(\beta,r)V_k(\beta,r)rdr. \tag{4}$$

Functions $V_i(\beta,r)=Y_i(\beta,R)J_i(\beta,r)+J_i(\beta,R)\times\times Y_i(\beta,r)$ are self-functions of boundary problems for variable r, $J_i(\beta,r)$, $Y_i(\beta,r)$ are Bessel functions of the first and second kind respectively.

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As is known, the calculation values of the integrals of the product of the Bessel functions of the first and second kind, the more the product of functions (4), except for the integrals of the squares of these functions for matching its own values β in the literature as a direct calculation of such integrals through the ranks, which represents the function does not work because of too much error when truncating these series end sequences.

The purpose of this work is to develop algorithms for the computation of integrals of the form indicated through their representation in the form of a finite sum of rational expressions of the second order. An apparatus that is used to represent a series of rational expressions are chain fractions [3].
Finite continued fraction can be written as

$$b_0 + a_1 b_1 + a_2 b_2 + \cdots + a_n b_n. \qquad (1)$$

Any continued fraction of the form (5), obtained as a result of a finite number of rational operations on its elements, is a rational function of elements and can be represented as a quotient of two polynomials

$$\frac{P_n(a_1, a_2, \ldots, a_n; b_0, b_1, \ldots, b_n)}{Q_n(a_1, a_2, \ldots, a_n; b_0, b_1, \ldots, b_n)},$$

which is called a convergent continued fraction. The numerator and denominator of the fraction are determined by recurrence expression

$$\begin{aligned} &P_n = b_n P_{n-1} + a_n P_{n-2}; \\ &Q_n = b_n Q_{n-1} + a_n Q_{n-2}; \quad n = 1, 2, \ldots \end{aligned} \qquad (2)$$

The difference between adjacent convergent

$$\frac{P_n}{Q_n} - \frac{P_{n-1}}{Q_{n-1}} = \prod_{i=1}^{n} a_i \frac{(-1)^{n+1}}{(Q_{n-1} Q_n)}, \; n = 1, 2, \ldots \qquad (3)$$

Consider the decomposition of the rational function (series) into a continued fraction.

$$\begin{aligned} &f(x) = \frac{\alpha_{10} + \alpha_{11}x + \alpha_{12}x^2 + \ldots}{\alpha_{00} + \alpha_{01}x + \alpha_{02}x^2 + \ldots} = \frac{\alpha_{10}}{\alpha_{00}} + \frac{\alpha_{20}x}{\alpha_{10}} + \frac{\alpha_{30}x}{\alpha_{20}} + \ldots, \\ &\alpha_{ik} = \alpha_{i-1,0}\alpha_{i-2,k+1} - \alpha_{i-2,0}\alpha_{i-1,k+1}; \; i = \overline{1,2n}; \; k = \overline{1,n} \end{aligned} \qquad (8)$$

whith $\alpha_{k,0} \neq 0; \quad \alpha_{0,0} = 1$.

Consider instead the fraction (5) corresponding to it suitable fraction.

$$f_1(x) = \frac{P_n(x)}{Q_n(x)}. \qquad (4)$$

The appropriate fraction (9) approximates the continued fraction (5) with an error

$$| f(x) - f_1(x) | \leq \frac{1}{\sqrt{5Q_n(x)}}.$$

The function can be written in this form

$$f_1(x) = \frac{\sum_{i=0}^{s} d_i x^i}{\sum_{j=0}^{r} c_j x^j}, \quad s < r, \quad c_0 = 1. \qquad (5)$$

The algorithm implementation

The goal of the algorithm of approximation functions by rational expressions is to calculate coefficients d_i, c_j ; $i = \overline{0,s}$; $j = \overline{0,r}$; by the following formulas [3].

Calculation of the coefficients of continued fractions.

$$\alpha_{k,0} \neq 0; \quad \alpha_{0.0} = 1; \quad i = \overline{1,n}; \quad k = \overline{1,n};$$

$$e_1 = \alpha_{1,0}\alpha_{0,0}; e_{i+1} = \alpha_{i+1,0}\alpha_{i-1,0}\alpha_{i,0}; \quad i = \overline{1,n-1};$$

$$\alpha_{ik} = \alpha_{i-1,0}\alpha_{i-2,k+1} - \alpha_{i-2,0}\alpha_{i-1,k+1}.$$

Calculation of the coefficients of the approximating a rational expression.

$$d_0 = b_0; \quad c_0 = a_0 = 1;$$

$$b_{2,2} = 0; \; b_{3,2} = d_{2,1}e_0; \; b_{i,1} = e_1; \; a_{i,1} = 1; \; i = \overline{1,n};$$

$$a_{2,2} = e_1; \quad a_{3,2} = e_1 + e_3;$$

$$b_{k,i} = e_k b_{k-2,i-1} + b_{k-1,1}; \quad k = \overline{3,m}; i = \overline{2,[k/2]};$$

$$a_{k,i} = e_k a_{k-2,i-1} + a_{k-1,1}; \quad k = \overline{3,m-1}; i = \overline{2,[k/2+1]};$$

$$d_i = b_{m,i}; \quad c_i = a_{m,i}; \quad i = \overline{1,n-1}.$$

With the proposed algorithm can be quite accurately approximated by fractional rational functions of high order to rational function of the second - third order. For Example,

$$\sum_{k=0}^{6} \frac{b_{2k}p + b_{2k+1}}{p^2 + a_{2k+1}p + a_{2k+2}} = \frac{d_2 p + d_1}{p^2 + c_1 p + c_0}.$$

IV. TASK SOLUTION

We approximate the rational expressions cylindrical function such as Bessel function. Bessel functions are written in the form

$$J_n(x)=\sum_{k=0}^{\infty}\frac{(-1)^k x/2^{n+2k}}{k!\Gamma(n+k+1)}. \qquad (6)$$

We pose the problem to achieve the accuracy of approximation. Imagine (11) as a finite continued fraction

$$J_n(x)=1+\frac{c_1^{(n)}}{1}+\frac{c_2^{(n)}}{1}+\ldots+\frac{c_m^{(n)}}{1}; \qquad (12)$$

$$J_n(x)\approx\sum_{k=0}^{5}\frac{b_{6k+3}+b_{6k+4}(x/2)}{b_{6k}+b_{6k+1}(x/2)+b_{6k+2}(x/2)^2}. \qquad (13)$$

For $J_0(x)$ we resive ($z=(x/2)^2$):

$$\bar{J}_0(z=(x/2)^2)\simeq\frac{1-3.2556z+2.2378z^2-0.54686z^3+0.054185z^4-0.0018907z^5}{1+0.24435z+0.0305z^2+0.00254z^3+10^{-5}(15.23z^4+0.642z^5+0.0154z^6)}. \qquad (14)$$

We transform this expression in the amount of units of the second order, calculating the roots of the denominator. We have

$$J_0(z)\simeq\bar{J}_0(z)=\frac{679310+65794z}{179.26+16.793z+z^2}$$
$$+\frac{-506130-79342z}{164.03+24.529z+z^2}+\frac{-154720+1310z}{220.11+0.24981z+z^2}.$$

Evaluation of error of approximation. $J_0(7)=0.30010;\ \bar{J}_0(7)=0.30010.$

Approximation of functions $J_1(x)$:

$$J_1(z)\simeq\frac{106790+16219z}{96.498+12.66z+z^2}$$
$$+\frac{-78570-18245z}{88.504+18.057z+z^2}+\frac{-25687-296.6z}{117.88+1.0346z+z^2}.$$

In a similar algorithm can be approximated by a rational expressions and Bessel functions of the second kind

$$Y_n(x)=\frac{2}{\pi}\big(C+\ln(x/2)\big)J_n(x)$$
$$-\frac{1}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{x}{2}\right)^{2k-n}$$
$$-\frac{1}{n}\sum_{k=0}^{n}\frac{(-1)^k}{k!(n+k)!}\left(\frac{x}{2}\right)^{2k+n}$$
$$\times\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k}+1+\frac{1}{2}+\cdots+\frac{1}{n+k}\right).$$

Since this expression is present, it must be well approximated by a rational function

$$\ln(x/2)\approx\frac{\sum_{k=0}^{m}c_{m+k}(x/2)^k}{\sum_{k=0}^{m}c_k(x/2)^k},\quad m=5;\quad m=6.$$

As a result of applying the algorithm above, the Bessel functions of the second kind obtain

$$Y_n(x)\approx\sum_{k=0}^{5}\frac{b_{6k+3}+b_{6k+4}(x/2)}{b_{6k}+b_{6k+1}(x/2)+b_{6k+2}(x/2)^2}. \qquad (7)$$

Using (13) and (15) the function $V_i(\beta,r)=Y_i(\beta,R)J_i(\beta,r)+J_i(\beta,R)Y_i(\beta,r)$ approximated by the following expression:

$$V_n(x)\approx\sum_{k=0}^{5}\frac{v_{6k+3}^{n,j}+v_{6k+4}^{n,j}(x/2)}{v_{6k}^{n,j}+v_{6k+1}^{n,j}(x/2)+v_{6k+2}^{n,j}(x/2)^2}. \qquad (8)$$

Coefficients v_k are depend from self-values β_{ij} of expressions $V_i(\beta,R)=Y_i(\beta,R)J_i(\beta,R)+J_i(\beta,R)\times$ $\times Y_i(\beta,R)=0.$

Now, the calculation of integrals of the form (4) is reduced to the calculation of integrals of rational functions of the form $f(x)=\dfrac{c_3+c_4x}{c_0+c_1x+c_2x^2}$, eg,

$$b_{1,2,3}^{1,2,3}=\int_{r_0}^{R}V_1(\beta_1,r)V_2(\beta_2,r)V_3(\beta_3,r)rdr$$
$$=\int_{r_0}^{R}\sum_{k=0}^{15}\frac{q_{6k+3}+q_{6k+4}r}{q_{6k}+q_{6k+1}r+q_{6k+2}r^2}rdr. \qquad (9)$$

These integrals can be easily calculated in analytic form. The implementation of these algorithms is executed on the PC in the algorithmic language.

Another group of algorithms associated with the calculation of integral transformations of variables φ and z is the axial component. Note that these algorithms for calculating integral transforms are realized once – before implementing the first iteration for the remaining iterations the values of integrals calculated before the implementation of the first iteration.

CONCLUSION

The problem with the calculation of integrals of the product of cylinder functions (in particular, the Bessel functions of 1st and 2nd kind) inevitably arises in the solution of nonlinear differential equations in partial derivatives). The traditional trip to the solution of this problem is the use of numerical methods that give satisfactory results only in special cases for one-dimensional problems. The proposed method of integrating the cylindrical functions can solve these problems with a prescribed accuracy and virtually in real time.

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Зеленська Н. К., Зеленський К. Х. Апроксимація функцій Бесселя дробово-раціональними функціями

Розглянуто задачу еквівалентного спрощення виразів, які отримують при розв'язанні системи квазілінійних рівнянь Нав'є–Стокса. Запропоновано алгоритм спрощення, що ґрунтується на використанні теорії ланцюгових дробів.

Ключові слова: інтегральні перетворення; ітераційна схема; конвективно-дифузійний перенос; ланцюгові дроби; рівняння Нав'є–Стокса; функції Бесселя; ланцюгові дроби.

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Зеленская Н. К., Зеленский К. Х. Аппроксимация функций Бесселя дробно-рациональными функциями

Рассмотрена задача эквивалентного упрощения функций Бесселя, получаемых при решении системы уравнений гидродинамики. Предложен алгоритм эквивалентного упрощения, основанный на применении теории цепных дробей.

Ключевые слова: интегральные преобразования; итерационная схема; конвективно-диффузионный перенос; уравнения Навье–Стокса; функции Бесселя; цепные дроби.

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