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ANALYSIS OF MATRIX ALGORITHM OF ATTITUDE DETERMINATION

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Abstract—*Relations which allow estimate the influence of vector measuring errors on matrix algorithm TRIAD accuracy are suggested. Connection between the algebraic and geometrical methods of errors analysis is achieved.*

Index terms—Orientation matrix; least squares method; errors.

I. Introduction

The most common way of orientation matrix $\hat{A}$ determination on the basis of information on vectors $\vec{b}_{o1}, \vec{b}_{o1},...,\vec{b}_{on}$ in reference coordinate system and vectors $\vec{b}_1, \vec{b}_1,...,\vec{b}_n$ in body coordinate system is the least squares method with minimization of losses function $l(A)$ [1]

$$l(A)=\frac{1}{2}\sum_{i=1}^{n}a_i\left\|\boldsymbol{b}_i - A\boldsymbol{b}_{oi}\right\|^2,$$

where a_i are weight coefficients.

The offered matrix method of this problem solution [2] gives the following expression of orientation matrix:

$$\hat{A}=D\left(\sqrt{D^T D}\right)^{-1}, \qquad (1)$$

where

$$D=MM_o^T;$$

$$M_o=\left[\boldsymbol{b}_{o1}^* \;\; \boldsymbol{b}_{o2}^* \; ... \; \boldsymbol{b}_{on}^*\right];\; M=\left[\boldsymbol{b}_1^* \;\; \boldsymbol{b}_1^* \; ... \; \boldsymbol{b}_1^*\right];$$

$$\boldsymbol{b}_{oi}^*=\sqrt{a_i}\boldsymbol{b}_{oi};\; \boldsymbol{b}_i^*=\sqrt{a_i}\boldsymbol{b}_i.$$

II. Problem formulation

Analyze the errors of orientation matrix $\hat{A}$ as a function of vectors $\boldsymbol{b}_i$ errors.

III. Solving the problem

At first let's consider the decision of task by the receipt of matrix of errors as a result of presence of errors of measuring of vectors. Let's write

$$M=AM_o+L,$$

where the matrix L characterizes error of measuring, which can be considered small.

Then

$$D=(AM_o+L)M_o^Y=AD_o+V,$$

where $D_o=M_oM_o^T$, $V=LM_o^T$.

Further let's write

$$D^TD=D_o^2+V^TAD_o{}^T+D_oA^TV+V^TV$$
$$\approx D_o^2+V^TAD_o{}^T+D_oA^TV.$$

To simplify the analysis let's consider that a matrix D_o is an identity one ($D_o=I$). Then

$$D^TD=I+V^TA+A^TV.$$

Let's use the next correlations (for small matrices X and S):

$$\sqrt{I+S}\approx I+\frac{1}{2}S;\; (I+S)^{-1}\approx I-S.$$

Taking into account these correlations we will have $\left(\sqrt{D^TD}\right)^{-1}=I-\frac{1}{2}\left(V^TA+A^TV\right)$. Then

$$\begin{aligned}\hat{A}&=D\left(\sqrt{D^TD}\right)^{-1}\\&=(A+V)\left(I-\frac{1}{2}\left(V^TA+A^TV\right)\right)\\&=\left(I+\frac{1}{2}\left(VA^T-AV^T\right)\right)A=\tilde{A}A,\end{aligned} \qquad (2)$$

where $\tilde{A}=I+B$; $B=\frac{1}{2}\left(VA^T-AV^T\right)$ is a matrix of errors, which is a small skew-symmetric matrix.

The matrix B can be written also in a form

$$\begin{aligned}B&=\frac{1}{2}\left(LM_o^TA^T-A\left(LM_o^T\right)^T\right)\\&=\frac{1}{2}\left(LM_1^T-M_1L^T\right),\end{aligned} \qquad (3)$$

or in a form

$$\begin{aligned}B&=\frac{1}{2}\left((M-M_1)M_1^T-M_1(M-M_1)^T\right)\\&=\frac{1}{2}\left(MM_1^T-M_1M^T\right).\end{aligned} \qquad (4)$$

where $M_1=AM_o$.

Thus, the presence of errors of measuring of vectors can be erected to introduction of additional imaginary turn, which is determined by the matrix of error B. Let's show the more simple method of matrix B determination. We will minimize the expression

$$r(B)=\frac{1}{2}tr\left[\left(M-\tilde{A}M_1\right)^T\left(M-\tilde{A}M_1\right)\right]-\frac{1}{2}tr\left[\Lambda\left(\tilde{A}^T\tilde{A}-I\right)\right]\approx\frac{1}{2}tr\left(L^TL-2BQ^T+B^TBD_1\right)-\frac{1}{2}tr\left[\Lambda\left(B+B^T\right)\right]. \quad (5)$$

where $Q=LM_1^T;\ D_1=M_1M_1^T$.

From a condition $\frac{\partial r}{\partial B}=0$ is found

$$B=\left[Q+\frac{1}{2}\left(\Lambda+\Lambda^T\right)\right]D_1^{-1}.$$

Here used relations $\frac{\partial XY^T}{\partial X}=Y;\ \frac{\partial X^TX}{\partial X}=2X.$ Accepting $D_1=I$, from a condition $B+B^T=0$ is found $\Lambda+\Lambda^T=-\left(Q+Q^T\right)$, i. e.

$$B=\frac{1}{2}\left(Q-Q^T\right)=\frac{1}{2}\left(LM_1^T-M_1L^T\right)=\frac{1}{2}\left(MM_1^T-M_1M^T\right). \quad (6)$$

This expression coincides with (4).

Let's consider the matrix B in detail. Its structure can be explained as follows. A direct solution of equation $L-BM_1=0$ is $B=LM_1^{-1}=LM_1^T$. Thus, this matrix, as any matrix of directed cosines for small angles, must be skew-symmetric. If a matrix $Z=XY^T$ is skew-symmetric, then $Z=-Z^T=-YX^T$. Therefore $Z+Z=XY^T-YX^T$, $Z=\frac{1}{2}\left(XY^T-YX^T\right)$.

The last expression gives a skew-symmetric matrix even if a matrix XY^T is not a skew-symmetric. Therefore a matrix $B=LM_1^T$ can be not skew-symmetric, but a matrix (3) is skew-symmetric.

Let's analyze the obtained expressions. Figure 1shows a transition from the coordinate system $X_oY_oZ_o$ into coordinate system XYZ.

Using relation

$$B_1=A^{-1}BA, \quad (7)$$

write the matrix B_1 in the reference coordinate system.

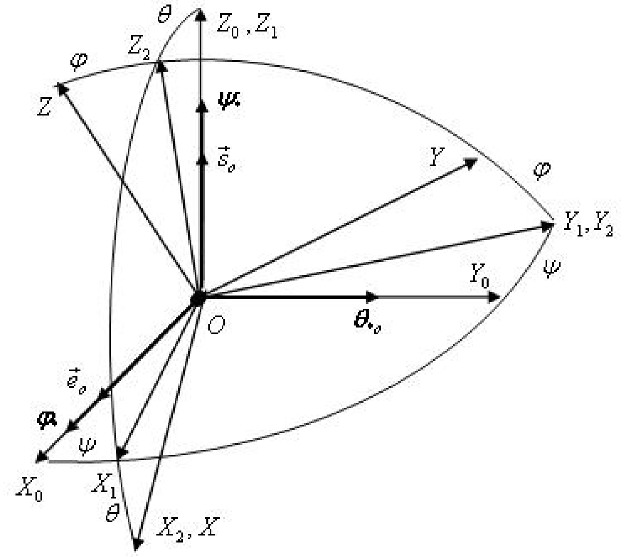

Fig. 1. Representing angle as a vector

The matrix B can be written in a form $B=\frac{1}{2}A^T\left(LM_1^T-M_1L^T\right)A.$ As $L=AL_o$, we will have

$$B_1=\frac{1}{2}A^T\left(AL_oM_o^{'}A^T-AM_oL_o^{'}A^T\right)A=\frac{1}{2}\left(L_oM_o^T-M_oL_o^T\right). \quad (8)$$

Let's consider the example. Suppose that there is a set of two reference vector $e_o=[1\ 0\ 0]'$, $s_o=[0\ 0\ 1]'$ and angles $\psi=10°;\ \theta=20°;\ \varphi=40°$. Error of measuring will set the rotation of vector $\vec{e}_n$ around the axis OY on a angle $1°$. Matrix B is calculated by (4), and matrix B_1 – by (7). The result of calculations of matrices is the following (values over are given in angular minutes)

$$\text{B}=\begin{bmatrix}0 & -35{,}2498 & -47{,}5525\\ 35{,}2498 & 0 & 9{,}7901\\ 47{,}5525 & -9{,}7901 & 0\end{bmatrix};$$

$$\text{B}_1=\begin{bmatrix}0 & 0 & -59{,}9970\\ 0 & 0 & 0\\ 59{,}997 & 0 & 0\end{bmatrix}.$$

The following skew-symmetric matrix corresponds to matrix B_1::

$$\tilde{B}_1=\begin{bmatrix}0 & \psi_* & -\theta_*\\ -\psi_* & 0 & \varphi_*\\ \theta_* & -\varphi_* & 0\end{bmatrix}, \quad (9)$$

where ψ_*, θ_*, φ_* - the angles of errors.

Let's represent an angle θ_{*_o} as a vector $\boldsymbol{\theta}_{*_o}$ (see Fig. 1). Elements of matrix B can be found as projections of vector $\boldsymbol{\theta}_{*_o}$ on axes of the body coordinate system. For example, $B(1,2)$ is the projection of vector $\boldsymbol{\theta}_{*_o}$ on an axis OZ:

$$B(1,2)=A(3,2)\theta_* = -35.2516'.$$

Let's find the errors of orientation angle determination using the matrix of error

$$\Delta\hat{A} = C, \tag{10}$$

where $C = AB_1$.

Let's find an expression for error $\Delta\psi$. Assume that the element $A(1,1) = \cos\psi\cos\theta$ of matrix A. We get:

$$\Delta A(1,1) = C(1,1), \tag{11}$$

where

$$\Delta A(1,1) = \frac{\partial A(1,1)}{\partial\psi}\Delta\psi + \frac{\partial A(1,1)}{\partial\theta}\Delta\theta = -\sin\psi\cos\theta\,\Delta\psi - \cos\psi\sin\theta\,\Delta\theta,$$

$\Delta\psi$, $\Delta\theta$ are errors of angle determination.

$$C(1,1) = A(1,1)B_1(1,1) + A(1,2)B_1(2,1) + A(1,3)B_1(3,1) = -\sin\psi\cos\theta\,\psi_{*_o} - \sin\theta\,\theta_{*_o}.$$

Let's consider the element $A(1,2) = \sin\psi\cos\theta$ of matrix A. We will have

$$\Delta A(1,2) = C(1,2), \tag{12}$$

where

$$\Delta A(1,2) = \frac{\partial A(1,2)}{\partial\psi}\Delta\psi + \frac{\partial A(1,3)}{\partial\theta}\Delta\theta = \cos\psi\cos\theta\,\Delta\psi - \sin\psi\sin\theta\,\Delta\theta;$$

$$C(1,2) = A(1,1)B_1(1,2) + A(1,2)B_1(2,2) + A(1,3)B_1(3,2) = \cos\psi\cos\theta\,\psi_{*_o} + \sin\theta\,\varphi_{*_o}.$$

Examining together

$$\sin\psi\cos\theta\,\Delta\psi + \cos\psi\sin\theta\,\Delta\theta = -C(1,1);$$
$$\cos\psi\cos\theta\,\Delta\psi - \sin\psi\sin\theta\,\Delta\theta = C(1,2),$$

we find

$$\Delta\psi = \frac{-C(1,1)\sin\psi}{\cos\theta} + \frac{C(1,2)\cos\psi}{\cos\theta} = \frac{\psi_{*_o}\cos\theta}{\cos\theta} + \frac{\sin\theta\left(\theta_{*_o}\sin\psi + \varphi_{*_o}\cos\psi\right)}{\cos\theta}. \tag{13}$$

Let's find an expression for an error $\Delta\theta$. Let's consider the element $A(1,3) = -\sin\theta$ of matrix A. We get

$$\Delta A(1,3) = C(1,3), \tag{14}$$

where $\Delta A(1,3) = \dfrac{\partial A(1,3)}{\partial\theta}\Delta\theta = -\cos\theta\,\Delta\theta.$

$$C(1,3) = A(1,1)B_1(1,3) + A(1,2)B_1(2,3) + A(1,3)B_1(3,3) = -\cos\psi\cos\theta\,\theta_{*_o} + \cos\theta\sin\psi\,\varphi_{*_o}.$$

$$\Delta\theta = \theta_*\cos\psi - \varphi_*\sin\psi. \tag{15}$$

Let's find an expression for an error $\Delta\varphi$. Assume that the element $A(2,3) = \cos\theta\sin\varphi$ of matrix A. We have

$$\Delta A(2,3) = C(2,3), \tag{16}$$

where

$$\Delta A(2,3) = \frac{\partial A(2,3)}{\partial\theta}\Delta\theta + \frac{\partial A(2,3)}{\partial\varphi}\Delta\varphi = -\sin\theta\sin\varphi\,\Delta\theta + \cos\theta\cos\varphi\,\Delta\varphi.$$

$$C(2,3) = A(2,1)B_1(1,3) + A(2,2)B_1(2,3) + A(2,3)B_1(3,3) = -\left(-\cos\varphi\sin\psi + \sin\varphi\cos\psi\sin\theta\right)\theta_{*_o} + \left(\cos\varphi\cos\psi + \sin\varphi\sin\psi\sin\theta\right)\varphi_{*_o}.$$

Let's consider the element $A(3,3) = \cos\theta\cos\varphi$ of matrix A. We have

$$\Delta A(3,3) = C(3,3),, \tag{17}$$

where

$$\Delta A(3,3) = \frac{\partial A(3,3)}{\partial\theta}\Delta\theta + \frac{\partial A(3,3)}{\partial\varphi}\Delta\varphi = -\sin\theta\cos\varphi\,\Delta\theta - \cos\theta\sin\varphi\,\Delta\varphi,$$

$$C(3,3) = A(3,1)B_1(1,3) + A(3,2)B_1(2,3) + A(3,3)B_1(3,3) = -\left(\sin\varphi\sin\psi + \cos\varphi\cos\psi\sin\theta\right)\theta_{*_o} + \left(-\sin\varphi\cos\psi + \cos\varphi\sin\psi\sin\theta\right)\varphi_{*_o}.$$

Examining together

$$-\sin\theta\sin\varphi\,\Delta\theta + \cos\theta\cos\varphi\,\Delta\varphi = C(2,3);$$
$$-\sin\theta\cos\varphi\,\Delta\theta - \cos\theta\sin\varphi\,\Delta\varphi = C(3,3),$$

we find

$$\Delta\varphi = \frac{1}{\cos\theta}\left(C(2,3)\cos\varphi - C(3,3)\sin\varphi\right) = \frac{1}{\cos\theta}\left(\theta_{*_o}\sin\psi + \varphi_{*_o}\cos\psi\right). \tag{18}$$

Let's specify vectors $\boldsymbol{\psi}_{*_o}, \boldsymbol{\theta}_{*_o}, \boldsymbol{\varphi}_{*_o}$ in the reference coordinate system (Fig. 2). Angles Δ_ψ, Δ_θ, Δ_φ also will be represented as vectors $\boldsymbol{\Delta}_\psi$, $\boldsymbol{\Delta}_\theta$, $\boldsymbol{\Delta}_\varphi$. The next relation takes place:

$$\psi_{*o} + \theta_{*o} + \varphi_{*o} = \Delta_\psi + \Delta_\theta + \Delta_\varphi. \qquad (19)$$

Project this expression on axes designated by stroke lines in Fig.2. They are the lines of intersections of rotation angle planes. The feature of these axes is that the only one of vectors Δ_ψ, Δ_θ, Δ_φ projects on each of them.

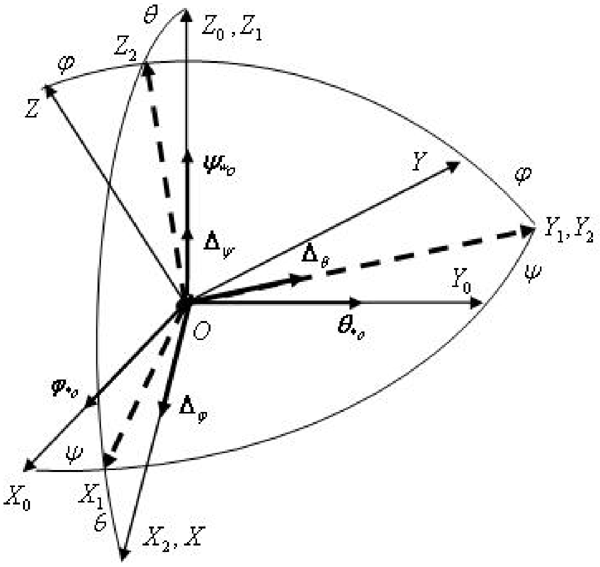

Fig. 2. Angles in the reference coordinate system

In a projection on an axis OZ_2 we get

$$\Delta\psi\cos\theta = \psi_{*o}\cos\theta + \sin\theta\left(\theta_{*o}\sin\psi + \varphi_{*o}\cos\psi\right),$$

i. e. (13).

In a projection on an axis OY_2 we get

$$\Delta_\theta = \theta_{*o}\cos\varphi - \psi_{*o}\sin\varphi,$$

i. e. (15).

In a projection on an axis OX_1 we get

$$\Delta_\varphi\cos\theta = \theta_{*o}\sin\psi + \varphi_{*o}\cos\psi,$$

i. e. (18).

Conclusion

Relations which allow estimate the influence of vector measuring errors on matrix algorithm TRIAD accuracy are suggested. Connection between the algebraic and geometrical methods of errors analysis is achieved.

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Л. М. Рижков, М. В.Ожог. Аналіз матричного алгоритму визначення орієнтації

Описано вплив похибок вимірювання опорних векторів на похибку визначення кутової орієнтації алгоритмом TRIAD. Показано взаємозв'язок між алгебраїчними і геометричними способами аналізу похибок.

Ключові слова: матриця орієнтації; метод найменших квадратів; похибки.

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Л. М.Рыжков, Н. В.Ожог. Анализ матричного алгоритма определения ориентации

Описано влияние погрешностей измерения опорных векторов на погрешность определения угловой ориентации алгоритмом TRIAD. Показана взаимосвязь между алгебраическими и геометрическими способами анализа погрешностей.

Ключевые слова: матрица ориентации; метод наименьших квадратов; погрешности.

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