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COMPARISON OF THE COVARIANCE ANALYSIS RESULTS OF THE QUEST AND THE TRIAD METHODS

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Abstract—*This paper describes the analytical expression of the attitude covariance matrix for the QUEST and TRIAD methods in an individual case. Influence of measurement model parameters on the accuracy of attitude determination via the methods method is analyzed. An influence of changes in the accuracy of one of the sensors on the attitude determination is considered.*

Index terms—Attitude determination method; QUEST; TRIAD; measurement model; covariance analysis; attitude covariance matrix.

I. INTRODUCTION

Deterministic attitude determination methods are based on the measurement of two or more base directions to some observed objects in a single point in time. These directions are known in the reference frame. In the body frame they are measured by the appropriate sensors. The case when only two base vectors are measured is common for such type of satellites as microsatellites. The two vectors are typically the unit vector to the Sun and the Earth's magnetic field vector for coarse "sun-mag" attitude determination or unit vectors to two stars tracked by two star trackers for fine attitude determination.

TRIAD was the first method to obtain three-axis attitude for spacecraft. Because its simplicity it has become one of the most popular ones [1]. In 1965 Grace Wahba proposed an attitude determination problem [2]. The problem is to find best estimate of the attitude matrix A based on a least squares criterion, i. e. to find the orthogonal matrix A with determinant +1 that minimizes the loss function

$$L(A)=\frac{1}{2}\sum_{i=1}^{n}a_i\left|\vec{b}_i-A\vec{r}_i\right|^2, \qquad (1)$$

where $\vec{b}_i$ vector measurements in the spacecraft body frame and $\vec{r}_i$ vectors in the reference frame; a_i is a set of positive weights assigned to each measurement. It was proven that the loss function can be rewritten as

$$L(A)=\lambda_0-tr\left(AB^T\right), \qquad (2)$$

with

$$\lambda_0=\sum_{i=1}^{n}a_i \text{ and } B=\sum_{i=1}^{n}a_i\vec{b}_i\vec{r}_i^{\,T}.. \qquad (3)$$

The loss function will be minimum when the trace of the matrix product AB^T is maximum, under constraint

$$A^TA=I. \qquad (4)$$

It can be seen that such problem formulation allows incorporating more that two measurements for the estimation of an attitude matrix. Moreover, measurements derived by means of different sensors are taken into account in different way through the coefficients a_i. Almost all deterministic methods solve the Wahba's problem in one way or another [3]. The QUEST is one of the most popular one. It is widely used in different mission [4].

In addition to attitude determination it is important to evaluate the accuracy of attitude determination. Covariance analysis is a common method used to solve this problem. It allows studying the relationship between errors in measurements and error in quantities derived from the measurements. An attitude covariance matrix is a statistical measure of the attitude estimation error. The results of covariance analysis of concerned methods are presented in [1].

II. PROBLEM FORMULATION

An accuracy of sensors is not constant and varies during the flight under the influence of various factors. The main goal of the article is to analyze the influence of the measurement model parameters on the attitude determination accuracy. The case when only two base vectors are measured is considered. The QUEST and TRIAD methods is used to attitude determination.

III. THE SOLUTIONS OF THE WAHBA'S PROBLEM (QUEST AND TRIAD)

Many attitude determination methods which solve Wahba's problem have been developed, but the first useful solution of the problem for spacecraft attitude determination was provided by Davenport [3], [4]. The method developed by him is known as

q-method. Davenport restated the problem (1) in terms of the quaternion $\bar{q} = \left[\vec{q}^T, q_4\right]^T$, for which

$$A = \left(q_4^2 - \vec{q}^T\vec{q}\right)I + 2\vec{q}\vec{q}^T - 2q_4\left[q\times\right], \tag{5}$$

where

$$\left[q\times\right] = \begin{bmatrix} 0 & q_3 & -q_2 \\ -q_3 & 0 & q_1 \\ q_2 & -q_1 & 0 \end{bmatrix}, \tag{6}$$

is skew-symmetric matrix.

Thus substitution of the quaternion instead of the rotation matrix leads to the next form of the loss function

$$g\left(\vec{q}\right) = \vec{q}^T K \vec{q}, \tag{7}$$

where K is the symmetric 4×4 matrix given by

$$K = \begin{bmatrix} S - \sigma I & Z \\ Z & \sigma \end{bmatrix} \tag{8}$$

with

$$S = B + B^T, \tag{9}$$

$$Z = \left[B_{23} - B_{32}, B_{31} - B_{13}, B_{12} - B_{21}\right]^T, \tag{10}$$

$$\sigma = tr\left(B\right). \tag{11}$$

To maximize the gain function (7), we take the derivative with respect to $\bar{q}$, but since the quaternion elements are not independent the constraint must also be satisfied. Adding the constraint $\bar{q}^T\bar{q} = 1$ to the gain function with a Lagrange multiplier yields a new gain function,

$$g'\left(\bar{q}\right) = \bar{q}^T K \bar{q} - \lambda\bar{q}^T\bar{q}. \tag{12}$$

Differentiating this gain function shows that $g'(\bar{q})$ has a stationary value when

$$K\bar{q} = \lambda\bar{q}. \tag{13}$$

The largest eigenvalue of K maximizes the gain function. The eigenvector corresponding to this largest eigenvalue is the least-squares optimal estimate of the attitude.

Q-method offers simple and elegant solution of the attitude estimation problem. But in this method you need to compute eigenvalues and eigenvectors of matrix K that is 4×4. At the time of its introduction it was a problem, because computing capabilities of onboard computers and even ground stations of monitoring and support were limited. Shuster introduced a method which estimates an attitude of a spacecraft less accurately then q-method, but it requires significantly fewer computations [1]. This contributed to its widespread use for attitude determination of spacecrafts and other types of moving objects. In accordance to the method the optimal quaternion is determined as follows

$$q_{\text{opt}} = \frac{1}{\sqrt{\vec{x}^2 + \gamma^2}}\begin{pmatrix} \vec{x} \\ \gamma \end{pmatrix}, \tag{14}$$

where

$$\vec{x} = \left[\alpha I + \left(\lambda_{\max} - tr\left(B\right)\right)S + S^2\right]z_v, \tag{15}$$

$$\gamma = \alpha\left[\lambda_{\max} + tr\left(B\right)\right] - \det\left(S\right), \tag{16}$$

and

$$\alpha = \lambda_{\max}^2 - \left[tr\left(B\right)\right]^2 + tr\left(adj\left(S\right)\right). \tag{17}$$

In the last expression adj means an adjoint matrix. The more detailed description of the method is given in [1] and [3].

The TRIAD is the simplest deterministic way to find the attitude matrix [1]. In accordance with it triads of orthonormal unit vectors are constructed in the reference frame and in the body frame:

$$\vec{v}_1 = \vec{r}_1, \vec{v}_2 = \frac{\vec{r}_1 \times \vec{r}_2}{\left|\vec{r}_1 \times \vec{r}_2\right|}, \vec{v}_3 = \vec{v}_1 \times \vec{v}_2, \tag{18}$$

$$\vec{w}_1 = \vec{b}_1,\ \vec{w}_2 = \frac{\vec{b}_1 \times \vec{b}_2}{\left|\vec{b}_1 \times \vec{b}_2\right|},\ \vec{w}_3 = \vec{w}_1 \times \vec{w}_2.. \tag{19}$$

Then based on constructed triads two matrices M_0 and M are composed:

$$M_0 = \left[\vec{v}_1 \mid \vec{v}_2 \mid \vec{v}_3\right], \tag{20}$$

$$M = \left[\vec{w}_1 \mid \vec{w}_2 \mid \vec{w}_3\right]. \tag{21}$$

As components of matrices M_0 and M are orthogonal unit vectors so these matrices are orthogonal matrices. The attitude matrix based on these matrices can be written as shown

$$A = MM_0^{-1} = MM_0^T. \tag{22}$$

It should be noted that as the first vector of triad $\vec{b}_1$ (and $\vec{r}_1$ respectively) should be selected a vector that is measured in body frame more precisely. Based on the attitude matrix A angles of rotation can be calculated if a sequence of rotations is known.

IV. MEASUREMENT MODEL AND COVARIANCE ANALYSIS

The attitude determination error covariance matrix or simply attitude covariance matrix is defined as follows:

$$P_{\theta\theta} = M\left[\Delta\vec{\theta}\Delta\vec{\theta}^T\right] = \begin{bmatrix} p_{xx} & p_{xy} & p_{xz} \\ p_{xy} & p_{yy} & p_{yz} \\ p_{xz} & p_{yz} & p_{zz} \end{bmatrix}, \quad (23)$$

where

$$\Delta\vec{\theta} = \left(\Delta\theta_1, \Delta\theta_2, \Delta\theta_3\right)^T, \quad (24)$$

is the error angle vector that is defined as the set of angles (measured in radians) of the small rotation carrying the true attitude matrix into the measured attitude matrix. The angles $\Delta\theta_i$ are defined in the body frame. This simplifies the calculation of the covariance matrix as compared with the case when attitude parameters are used. The covariance matrix $P_{\theta\theta}$ is positive definite and symmetric, $P_{\theta\theta} = P_{\theta\theta}^T > 0$ which means that there are only six unique elements in the matrix.

The attitude covariance matrix for the QUEST method is defined as follows [1], [6]:

$$P_{\theta\theta}^{QU} = \left[\sum_{i=1}^{n} \frac{1}{\sigma_i^2}\left(I - \vec{b}_i\vec{b}_i^T\right)\right]^{-1}. \quad (25)$$

The attitude covariance matrix for the TRIAD method is defined as follows [6]:

$$P_{\theta\theta}^{TR} = \frac{1}{\left|\vec{w}_2'\right|^2}\left[\sigma_1^2\left(\vec{w}_2\vec{w}_2^T + \vec{w}_2'\left(\vec{w}_2'\right)^T\right) + \sigma_2^2\vec{w}_1\vec{w}_1^T\right], \quad (26)$$

where $\vec{w}_2'$ is unnormalized vector $\vec{w}_2$ defined in (19). It should be noted that in practice the measured vectors $\vec{b}_i^*$ are used here instead of true values $\vec{b}_i$ because the last ones are unknown.

The article [1] notes that matrix $P_{\theta\theta}$ is attitude independent while the same matrix significantly depends on the attitude when it is recorded in the parameters of attitude which describe orientation of the body frame with respect to the reference frame.

The measurement model is a set of equations used to analyze an accuracy of attitude estimation derived by one or another method. We use the QUEST measurement model (QMM) [6], [7] which is quite common. According to the author of the model, it is pretty simple, but realistic.

QUEST measurement model is based on the assumption that measured vectors $\vec{b}_i^*$ with high probability are concentrated in a small volume about the true direction [6]. In accordance to QMM,

$$\vec{b}_i^* = A\vec{r}_i + \Delta\vec{b}_i, \quad (27)$$

$$\Delta\vec{b}_i A\vec{r}_i = 0, \quad (28)$$

where $\Delta\vec{b}_i$ - the vectors of measurement errors of i-th vector of reference direction. As you can see from (28), vectors $\Delta\vec{b}_i$ are in the plane that is perpendicular to the true direction $A\vec{r}_i$ in the body frame. This plane is an approximation of the part of the sphere that contains ends of vectors $A\vec{r}_i$ and $\vec{b}_i^*$. The distribution of the components of the i th measurement error vector perpendicular to the true are assumed to be Gaussian and uniformly distributed in phase about the true vector with variance σ_i^2 per axis. The vectors $\Delta\vec{b}_i$ have such properties:

$$M\left[\Delta\vec{b}_i\right] = \vec{0}, \quad (29)$$

$$M\left[\Delta\vec{b}_i\Delta\vec{b}_i^T\right] = \sigma_i^2\left[I - (A\vec{r}_i)(A\vec{r}_i)^T\right], \quad (30)$$

where σ_i is standard deviations of vector component values. These variances of unit vectors can be interpreted as angular variances in radians. Work [6] states that the error distribution $\Delta\vec{b}_i$ is axially symmetric about direction $A\vec{r}_i$.

Thus QMM which is described by the equations (27) – (30) defines a cone that can contain a measured vector $\vec{b}_i^*$ with some probability. In accordance to the measurement model the error vector rotates the true vector turning it to the measured vector. It should be noted that QMM is assumed that expected value of the measured vector is equal to zero, i. e. systematic errors are absent.

The weights a_i are chosen to be inverse variances [6] i. e.

$$a_i = \frac{1}{\sigma_i^2}. \quad (31)$$

V. Comparison of the Covariance Analysis Results

The square root of trace of the attitude covariance matrix is widely used as a scalar characteristic of attitude determination accuracy. Thus the attitude estimation error is characterized by the value:

$$\sigma_\theta = \sqrt{tr\left(P_{\theta\theta}\right)}. \quad (32)$$

Our work considers that the reference frame and the body frame are coinciding. Reference vectors in the reference frame directed along the coordinate axes for simplification i. e. $\vec{r}_1 = [1, 0, 0]^T$ and $\vec{r}_2 = [0, 1, 0]^T$. Considering that the reference frame and the body frame are coinciding, our simplification allows obtaining a simple analytic expression for the attitude covariance matrix (25). It will be as follows

$$P_{\theta\theta}^{QU} = diag\left(\sigma_2^2, \sigma_1^2, \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right). \quad (33)$$

Therefore from the formula (31) we obtain

$$\sigma_\theta^{QU}(\sigma_1,\sigma_2) = \sqrt{\sigma_1^2+\sigma_2^2+\frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}}. \quad (34)$$

For the TRIAD

$$\sigma_\theta^{TR}(\sigma_1,\sigma_2) = \sqrt{2\sigma_1^2+\sigma_2^2}. \quad (35)$$

Values of QMM parameters σ_i vary in the range from 0.001 to 0.05, step 0.0025. Then the surfaces σ_θ^{QU} and σ_θ^{TR} will be as shown on Figs 1 and 2.

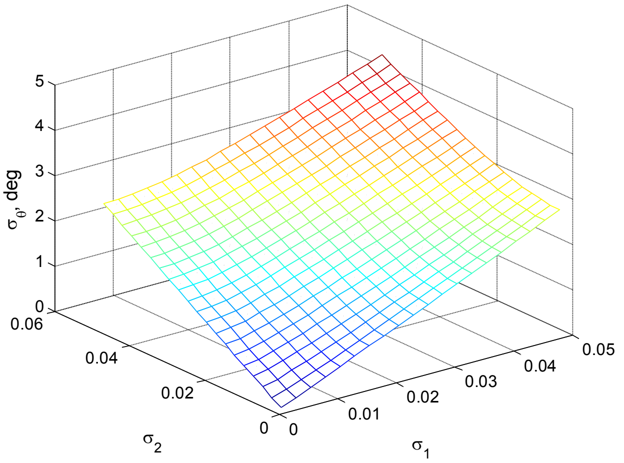

Fig. 1. Attitude error derived from the QUEST covariance matrix

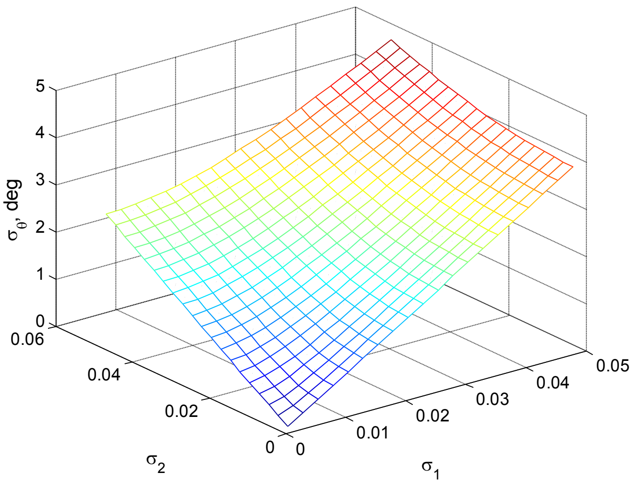

Fig. 2. Attitude error derived from the TRIAD covariance matrix

The accuracy of measurement of the first vector in (19) is more important for the TRIAD because of the triad is constructed on the basis of this vector. The advantage of the QUEST appears when more than two sensors are used. Herewith the influence of the least accurate sensor is slight. But using of the QUEST of itself do not increase significantly the attitude determination accuracy for the case of two vectors.

Numerical values of attitude determination error for the QUEST and TRIAD were also calculated. For each pair of values σ_1 and σ_2 $N = 1000$ tests were performed. The attitude determination error is measured by using the rotation angle between the true and estimated attitudes:

$$\varepsilon = \arccos\left(\frac{1}{2}\left(tr\left(\hat{A}^T\hat{A}\right)-1\right)\right). \quad (36)$$

The results of calculation in accordance to the (36) for both concerned methods match with the corresponding results derived analytically. If a sun sensor and a magnetometer are used to obtain attitude, then $\sigma_1 < \sigma_2$. The difference between the values of the error calculated from (35) and (34) is shown on Fig. 3.

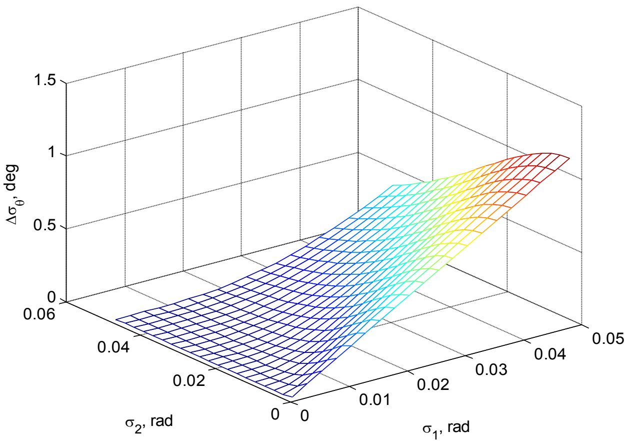

Fig. 3. Attitude error difference between the TRIAD and the QUEST methods

Conclusion

Usage of the covariance analysis allows estimating a probable error of attitude estimation caused by some types of sensor errors. The used measurement model allows taking into consideration white noise error of the measured vectors. The paper considers the case of measurement of two reference vectors, which is the simplest, but very common, especially for microsatellites. The surfaces analytical expression which describes the dependence of the attitude determination error on the measurement model parameters were obtained for this case. The QUEST has a little advantage over the TRIAD when the accuracy of the first sensor (more accurate one) is not high. When the accuracy of the vector that is used to construct the triple of the auxiliary TRIAD vectors is decreased then advantage of the QUEST is increased.

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Л. М. Рижков, Д. І. Степуренко. Порівняння результатів коваріаційного аналізу методів QUEST та TRIAD

Представлено аналітичні вирази для коваріаційної матриці орієнтації, отримані для окремого випадку для методів QUEST та TRIAD. Проаналізовано вплив параметрів моделі вимірювань на точність визначення орієнтації за вказаними методами. Розглянуто питання впливу зміни точності одного з вимірювачів на точність визначення орієнтації.

Ключові слова: метод визначення орієнтації; QUEST; TRIAD; модель вимірювань; коваріаційний аналіз; коваріаційна матриця орієнтації.

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Л. М. Рыжков, Д. И. Степуренко. Сравнение результатов ковариационного анализа метод QUEST и TRIAD

Представлены аналитические выражения для ковариационной матрицы ориентации, полученные для частного случая для методов QUEST и TRIAD. Проанализировано влияние параметров модели измерений на точность определения ориентации с помощью указанных методов. Рассмотрен вопрос влияния изменения точности одного из измерителей на точность определения ориентации.

Ключевые слова: метод определения ориентации; QUEST; TRIAD; модель измерений; ковариационный анализ; ковариационная матрица ориентации.

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