AUTOMATIC CONTROL SYSTEMS

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STABLE LOW-ORDER CONTINUED FRACTION APPROXIMATIONS FOR SYSTEM IDENTIFICATION

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Abstract—*Algorithm for determination of transfer function models for transient processes from noisy data with continued fraction approximations is proposed. Presented method allows to solve the stability problem of low-order continued fraction approximations.*

Index Terms—System identification; low-order approximation; time-delay; continued fraction; Rutishauser's method; Shure's criterion.

I. INTRODUCTION

There has been always interest to continued fractions as to mathematical apparatus for approximations as it has many remarkable advantages. Among these advantages are following two 1) *n*th order continued fraction expansions are the best rational approximants, 2) this mathematical apparatus has quite good calculation stability, 3) calculation of coefficients of continued fractions can be effectively realized in microcontrollers [1], [2].

Using continued fractions for system identification was proposed in [3]. As it was shown in consequent papers, this prospective approach can be implemented for various identification problems, including parametrical and structural parametrical identification from transient and impulse responses for open loop systems and for closed loop ones [3]–[6].

Main problems in practical application of continued fractions for system identification are problems of stability and convergence [6]. These problems arise when experimental data is affected with noise of significantly high level as well as when order of continued fraction approximation is too low.

As models of low order are in particular situations more preferable than high-order ones and there are many situations when noise-level is very high, the problem of low-order approximation in system identification is actual problem of practical interest.

In this paper we propose algorithm for identification of time-delay systems from noisy transient responses using mathematical apparatus of continued fractions and special stabilization procedure.

II. MAIN RESULT

Identification process, according to the proposed approach, can be divided on 5 consequent stages:

1. Data acquisition.
2. Stabilization of experimental sequence.
3. Identification of discrete-time transfer function using continued fraction expansion.
4. Retrieving of continuous-time transfer function (if needed).
5. Adjustment of model's parameters.

When information on the technological process's parameters systems is introduced in a digital form (as output of analog-to-digital convertor), experimental data can be easily represented in a form of formal Laurent series:

$$f(z) = c_0 + c_1 z^{-1} + c_2 z^{-2} + c_3 z^{-3} \ldots, \qquad (1)$$

where $\{c_k\}_{k=0}^{m}$ is experimental output sequence, $c_k \in \mathbb{R}$; T is sampling period, *s*; $z = e^{Ts}$; $s = \sigma + j\omega$.

Continued fractions can be used for approximation of analytic functions. Continued fraction approximations take on values in the extended complex plane and may converge in regions that contain isolated singularities of the function to be represented.

Selection of an order of approximation is the problem to solve at first stage of identification.

Elements of the series (1) are related by recurrent equations determined by the system structure:

$$f(z) = \sum_{i=0}^{\infty} c_i z^{-i} = \frac{Y(z)}{X(z)} = \frac{\sum_{n=0}^{\infty} y(nT) z^{-nT}}{\sum_{n=0}^{\infty} x(nT) z^{-nT}},$$

where $X(z)$ is Z-image of input signal; $Y(z)$ is Z-image of output signal; $y(nT)$ is sequence of output values; $x(nT)$ is sequence of input values.

The variety of practical methods of representing power series in a continued fraction form as well as

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General theory of continued fractions conformity with formal Loran series are described in [7], [8].

The Rutishauser's method was chosen to be the method of representing analytic functions by continued fractions. This choice is conditioned by several advantages of using Rutishauser method as it was proved in [5].

Rutishauser's method is determined by the formula [7]:

$$f(z) \approx \frac{c_0}{1-}\frac{q_1^{(0)}z^{-1}}{1}\frac{e_1^{(0)}z^{-1}}{-\ 1}\frac{q_2^{(0)}z^{-1}}{-\ 1}\frac{e_2^{(0)}z^{-1}}{-\ 1}\ -\cdots$$

$$= \cfrac{c_0}{1-\cfrac{q_1^{(0)}z^{-1}}{1-\cfrac{e_1^{(0)}z^{-1}}{1-\cfrac{q_2^{(0)}z^{-1}}{\ddots\, 1-\cfrac{q_{n-1}^{(0)}z^{-1}}{1-e_{n-1}^{(0)}z^{-1}}}}}}$$

$$= \frac{P(c_0,\{e^{(0)}\},\{q^{(0)}\})}{Q(\{e^{(0)}\},\{q^{(0)}\})}, \qquad (2)$$

where $e_m^{(n)} \in \mathbb{C}$, $q_m^{(n)} \in \mathbb{C}$, $f(z) \in \hat{\mathbb{C}}$, $\hat{\mathbb{C}} = \mathbb{C} \cup [\infty]$; $\{e_m^{(n)}\}$ and $\{q_m^{(n)}\}$ are sequences determined with following relations: $e_m^{(n)} = q_m^{(n+1)} - q_m^{(n)} + q_{m-1}^{(n+1)}$; $q_{m+1}^{(n)} = \frac{e_m^{(n+1)}}{e_m^{(n)}} q_m^{(n+1)}$; $m = 1, 2, 3\ldots$; $n = 0, 1, 2, 3\ldots$; except $e_0^{(n)} = 0$ and $q_1^{(n)} = \frac{c_{n+1}}{c_n}$.

To determine the most proper order of approximation (which determines structure of the model) following methodic proposed: definition of the structure is made by dropping a number of indexes that correspond to values close to zero in the residual fraction numerator.

As it is implied that in (2) $e_m^{(n)} \neq 0$ and $q_m^{(n)} \neq 0$ (for $m = 1, 2, 3\ldots$), the sequence $\{c_n\}$ should be shifted to the first nonzero element and the resulting continued fraction should be multiplied by z^{-d} according to the delay theorem, where d is a shift of the lattice function.

Continued fraction coefficients can be expressed in terms of experimental sequence's elements $\{c\}$:

$$f(z) = c_0 + c_1 z^{-1} + c_2 z^{-2} + c_3 z^{-3}\ldots;\quad q_1^1(z) = \frac{c_2}{c_1}.$$

$$e_1^1(z) = \frac{c_3}{c_2} - \frac{c_2}{c_1};$$

$$q_1^2(z) = c_3\left(\frac{c_3}{c_2} - \frac{c_4}{c_3}\right) \Big/ c_2\left(\frac{c_2}{c_1} - \frac{c_3}{c_2}\right);$$

$$e_1^2(z) = \frac{c_4}{c_3} - \frac{c_3}{c_2} - \left(c_3\left(\frac{c_3}{c_2} - \frac{c_4}{c_3}\right)\right)\Big/\left(c_2\left(\frac{c_2}{c_1} - \frac{c_3}{c_2}\right)\right) + \left(c_4\left(\frac{c_4}{c_3} - \frac{c_5}{c_4}\right)\right)\Big/\left(c_3\left(\frac{c_3}{c_2} - \frac{c_4}{c_3}\right)\right);$$

$$q_1^2(z) = \frac{\left(-c_2^{\,2} + c_1c_3\right)}{\left(-c_3^{\,2} + c_2c_4\right)} \times \frac{\left(c_6c_3^{\,2} - 2c_3c_4c_5 + c_4^{\,3} - c_2c_6c_4 + c_2c_5^{\,2}\right)}{\left(c_5c_2^{\,2} - 2c_2c_3c_4 + c_3^{\,3} - c_1c_5c_3 + c_1c_4^{\,2}\right)}.$$

Corresponding continued fractions and characteristic polynomials:

$$f_1(z) = \frac{-c_0c_1z}{c_2 - c_1z};\quad D_1(z) = c_2 - c_1z;\quad f_2(z) = c_0 + \frac{c_0c_2^{\,2}}{c_1(c_3 - c_2z)};\quad D_2(z) = c_1c_3 - c_1c_2z;$$

$$f_3(z) = \frac{-\left(c_0z\left(c_4c_1^{\,2} + zc_1c_2^{\,2} + c_2^{\,3}\right) - c_0c_3z\left(zc_1^{\,2} + 2c_2c_1\right)\right)}{c_1\left(-c_2^{\,2}z^2 + c_2c_3z + c_4c_2 - c_3^{\,2} + c_1c_3z^2 - c_1c_4z\right)},$$

$$D_3(z) = c_1(-c_2^{\,2}z^2 + c_2c_3z + c_4c_2 - c_3^{\,2} + c_1c_3z^2 - c_1c_4z) = (c_1^{\,2}c_3 - c_1c_2^{\,2})z^2 + (c_1c_2c_3 - c_1^{\,2}c_4)z + c_1c_2c_4 - c_1c_3^{\,2};$$

$$f_4(z) = c_0 - \frac{c_0\left(c_5c_2^{\,2} + zc_2c_3^{\,2} + c_3^{\,3}\right) - c_0c_4\left(zc_2^{\,2} + 2c_3c_2\right)}{c_1\left(-c_3^{\,2}z^2 + c_3c_4z + c_5c_3 - c_4^{\,2} + c_2c_4z^2 - c_2c_5z\right)};$$

$$D_4(z) = c_1\left(-c_3^{\,2}z^2 + c_3c_4z + c_5c_3 - c_4^{\,2} + c_2c_4z^2 - c_2c_5z\right) = \left(c_1c_2c_4 - c_1c_3^{\,2}\right)z^2 + \left(c_1c_3c_4 - c_1c_2c_5\right)z + c_1c_3c_5 - c_1c_4^{\,2},$$

etc.

Discrete-time system is stable when all its poles are inside the unit disc. Such a way, when we have unstable continued fraction approximation our task is to correct its poles to "move" all of them into the unit disc (Fig. 1).

We propose correct not poles but the experimental sequence $\{c\}$ used for continued fraction expansion, that allows to achieve best fitness of approximation curve to experimental data as well as accelerate iterative computational procedure.

To fully automatize stability-correction, we propose to execute stability analysis with Shure's criterion (or other algebraic criterion) [9]: for characteristic polynomial of the discrete-time transfer function:

$$D(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n .$$

Consequent matrices

$$\Delta_k = \begin{vmatrix} a_0 & 0 & 0 & \cdots & 0 & a_n & a_{n-1} & \cdots & a_{b-k+1} \\ a_1 & a_0 & 0 & \cdots & 0 & 0 & a_n & \cdots & a_{b-k+2} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{k-1} & a_{k-2} & a_{k-3} & \cdots & a_0 & 0 & 0 & \cdots & a_n \\ a_n & 0 & 0 & \cdots & 0 & a_0 & a_1 & \cdots & a_{k-1} \\ a_{n-1} & a_n & 0 & \cdots & 0 & 0 & a_0 & \cdots & a_{k-2} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ & & & \cdots & & & & \cdots & \\ a_{n-k+1} & a_{n-k+2} & a_{n-k+3} & \cdots & a_n & 0 & 0 & \cdots & a_0 \end{vmatrix} .$$

should have number of sign changes equal to the order of characteristic polynomial, that's: $\Delta_1 < 0$, $\Delta_2 > 0$, $\Delta_3 < 0$, $\Delta_4 > 0$ … $(-1)^n \Delta_n > 0$.

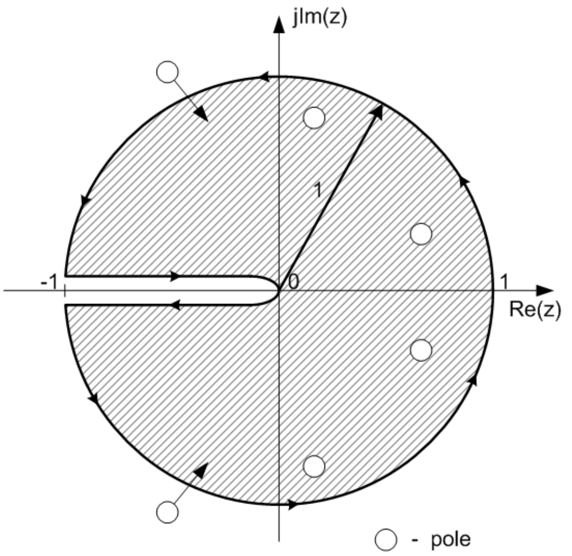

Fig. 1. *Z*-plane pole adjustment

For instance, for 3rd order continued fraction (which has 2nd order characteristic polynomial) we'll have:

$$\Delta_1 = \begin{vmatrix} a_0 & a_2 \\ a_2 & a_0 \end{vmatrix} = a_0^2 - a_2^2 < 0,$$

$$\Delta_2 = \begin{vmatrix} a_0 & 0 & a_2 & a_1 \\ a_1 & a_0 & 0 & a_2 \\ a_2 & 0 & a_0 & a_1 \\ a_1 & a_2 & 0 & a_0 \end{vmatrix}$$

$$= (a_0^2 - a_2^2)^2 - a_1^2 (a_0 - a_2)^2 > 0.$$

And Δ's can be expressed in terms of $\{c\}$:

$$\Delta_1 = \left(c_1 c_2 c_4 - c_1 c_3^2\right)^2 - \left(c_1^2 c_3 - c_1 c_2^2\right)^2 < 0,$$

$$\Delta_2 = \Delta_1^2 - \left(c_1 c_2 c_3 - c_1^2 c_4\right)^2$$
$$\times \left(c_1 c_2 c_4 - c_1 c_3^2 - c_1^2 c_3 + c_1 c_2^2\right)^2 > 0.$$

Such a way, stability conditions can be formulated as a system of n nonlinear inequations:

$$\begin{cases} \Delta_1(\{c\}_n) < 0; \\ \Delta_2(\{c\}_n) > 0; \\ \Delta_3(\{c\}_n) < 0; \\ \vdots \\ (-1)^n \Delta_n(\{c\}_n) > 0. \end{cases} \qquad (3)$$

Let's introduce corrected sequence $\{ \overline{c_i} \}$ that makes (3) held:

$$\overline{c_i} = c_i + d_i ,$$

where d_i are correcting differences (that assumed to be admissible small) that are defined based on absolute value of differences between current, next and previous values in experimental sequence.

Therefore, we can determine stable approximation with solution of optimization problem, where minimizing functional can be formulated as

$$J = \sum_{i=0}^{n} (\overline{c_i} - c_i)^2 \to \min .$$

Then, using $\{\overline{c}\}$ for continued fraction expansion we'll retrieve stable approximation of order n.

Let's illustrate this method with example.

Assume that we selected 3rd order continued fraction and we got following experimental output sequence (T = 1s): 0; 0; 0; 0.1420; 0.3801; 0.5805; 0.7410; 0.8399; 0.9; …

This experimental data gives us unstable transfer function as $\Delta_1 = 0.00010454 > 0$.

Simple iterative search in 60000 iterations in quite short value range gave us 3175 "stable" sequences and 56825 "unstable" sequences.

Examples of "stable" sequences are following:

1) 0.1420; 0.3785; 0.5855; 0.7380; 0.8419 …
2) 0.1420; 0.2801; 0.4805; 0.6730; 0.8559 …
3) 0.1420; 0.3121; 0.4965; 0.6410; 0.7399 …
4) 0.1420; 0.2785; 0.6835; 0.8360; 0.7419 …

…

As first in this list sequence is the closest one to experimental data so that it should be used for continuous fraction expansion and it'll give us discrete-time transfer function:

$$F(z) = z^{-2}\frac{0.142z + 0.09023}{z^2 - 1.03z + 0.2567}. \qquad (4)$$

Determination of model in a from of continuous transfer function, if needed, can be realized with inverse Z-transform. Minimal order continuous transfer function can be retrieved with using zero-order hold assumption or matched mapping between z-plane and s-plane.

During matched inverse Z-transform transformation should be carried out within the general frequency band, limited by Nyquist frequency according to the Nyquist-Shannon-Kotelnikov sampling theorem. On the basis of matched Z-transform properties negative roots of z-plane are looped off during the mapping to the s-plane [10].

Gain coefficient can be determined with formula

$$k = y_{ss}\frac{\prod |s_p|}{\prod |s_z|}.$$

where y_{ss} is steady-state value of the discrete transfer function ($y_{ss} = \lim_{z\to 1} G(z)$); s_p are poles of continuous transfer function; s_z is initial zero approximations.

If we apply Z-inverse transform for (4) we'll receive following model in a form of continuous-time transfer function:

$$F(s) = \frac{0.44e^{-2s}}{s^2 + 1.36s + 0.43}.$$

Experimental data and determined continuous -time transfer functions are shown in Fig. 2.

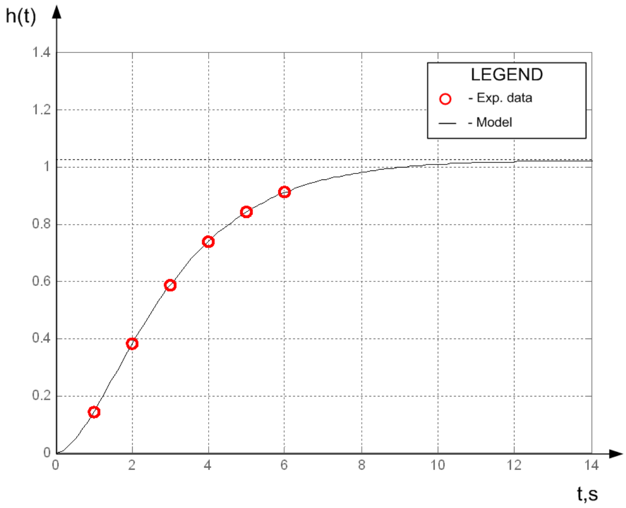

Fig. 2. Results of identification

As we can see in Fig. 2 model approximates experimental process with high accuracy.

In general case the time-delay of continuous object can be represented as: $\tau = \nu T - \Delta\tau$, where ν is time-delay samples; $\Delta\tau$ is time-delay sampling error, $0 \le \Delta\tau < T$. When $\Delta\tau \ne 0$ the shift of lattice function is present that causes the zeroes distortion of retrieving continuous transfer function. More accurate (appropriate) value of continuous time-delay can be determined with simple iterative procedure.

III. Summary

New algorithm for identification of systems from real transient responses was proposed. The method of identification allows to solve the stability problem of low-order continued fraction approximations in automatic mode (user should specify order of approximation only).

Efficiency of proposed method was demonstrated with short illustrative example.

Proposed method involves iterative procedure utilizing numerical methods, and as result of this it is difficult to apply this method for real-time identification.

As for future research this method should be optimized to provide higher speed of stable solution search.

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О. В. Глушко, Р. Ю. Ткачов. Стійкі апроксимації низького порядку для вирішення задачі ідентифікації

Розглянуто алгоритм визначення моделей у вигляді передатних функцій на базі зашумлених перехідних процесів, який базується на використанні апроксимацій ланцюговими дробами. Запропонований метод дозволяє вирішити проблему стійкості апроксимацій низького порядку у разі використання ланцюгових дробів.

Ключові слова: ідентифікація; апроксимація низького порядку; запізнювання у часі; ланцюговий дріб; метод Рутісхаузера; критерій Шура.

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О. В. Глушко, Р. Ю. Ткачев. Устойчивые аппроксимации низкого порядка для решения задачи идентификации

Рассмотрен алгоритм определения моделей в виде передаточных функций на основе зашумленных переходных процессов, основаный на использовании аппроксимаций цепными дробями. Предложенный метод позволяет решить проблему устойчивости аппрокцимаций низких порядков при использовании цепных дробей.

Ключевые слова: идентификация; апроксимация низкого порядка; задержка по времени; цепная дробь; метод Рутисхаузера; критерий Шура.

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