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DETERMINING SOME ELEMENTS OF ATTITUDE MATRIX

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Abstract. *A method for determining of some elements of the direction cosine matrix is proposed. A matrix form of solution of the aircraft attitude estimation problem based on using the least squares criterion is also presented. A problem of the sensitivity of attitude estimates to inaccuracies of input data is analyzed using QUEST measurement model.*

Keywords: attitude estimation method; matrix form of solution;Wahba's problem; least squares criterion; measurement model.

Introduction

The need of attitude determination has arisen almost simultaneously with the appearance of spacecrafts. The problem of finding the best estimate of the A matrix was posed by Grace Wahba [1] who was the first to choose a least squares criterion to define the best estimate, i.e. to find the orthogonal matrix **A** with determinant +1 that minimizes the loss functional

$$L(A) = \frac{1}{2}\sum_{i=1}^{n} c_i \left|\vec{b}_i - A\vec{r}_i\right|^2, \qquad (1)$$

where c_i are a set of positive weights assigned to each measurement; n is a number of vector measurements. The weights are selected as follows [3]

$$c_i = \frac{1}{\sigma_i^2}, \qquad (2)$$

where σ_i is a parameter of the measurement model which are characterized the accuracy of sensor i. At the same time the first solutions was proposed [2]. Since the problem has been formulated for the matrix of orientation the solutions were obtained for this matrix. However, at the time of the appearance they had some limitations that did not allow them to be implemented in practice. As a result, more efficient methods have been developed that used the quaternion representation of orientation [3], [4].

Problem formulation

Most of the existed attitude determination methods determine a quaternion an attitude matrix. But use Euler angles in the analysis conveniently as they have simple geometric interpretation. This is especially relevant for such class of moving objects as spacecrafts. We propose to determine only those elements of attitude matrix which are needed to calculate Euler angles based on the accepted sequence of rotations.

Matrix method for solving the problem of estimating

Under ideal conditions when measurement errors and calculation ones are absent the next equation holds

$$\vec{b}_i = A\vec{r}_i\,, \qquad (3)$$

where $\vec{r}_i$ and $\vec{b}_i$ are unit vectors of base directions in the reference frame and body frame respectively; A is attitude matrix which moves reference frame to the body frame. Equation (3) can be rewritten in matrix form as follows

$$\mathbf{M} = A\mathbf{M}_0\,,$$

where $\mathbf{M} = \left[\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\right]$ and $\mathbf{M}_0 = \left[\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n\right]$ are matrices constructed from unit vectors. Since, in general, the matrices are rectangular the attitude matrix is determined as follows

$$A = \mathbf{M}\mathbf{M}_0^{\mathrm{T}}\left(\mathbf{M}_0\mathbf{M}_0^{\mathrm{T}}\right)^{-1}. \qquad (4)$$

After transposition we have

$$A^{\mathrm{T}} = \left(\mathbf{M}_0\mathbf{M}_0^{\mathrm{T}}\right)^{-1}\mathbf{M}_0\mathbf{M}^{\mathrm{T}}\,, \qquad (5)$$

It should be noticed that solutions (4) and (5) exist subject to at least three base directions are observed. In the other case $\mathbf{M}_0\mathbf{M}_0^{\mathrm{T}}$ is a singular matrix.

In real case do to the presence of errors in vectors $\vec{b}_i$ the attitude matrix derived by the (4) or (5) is not orthogonal. For this matrix it is necessary to perform the operation of orthogonalization. This can be done using the following iterative formula [5]

$$A_{n+1} = \frac{3}{2}A_n - \frac{1}{2}A_n A_n^{\mathrm{T}} A_n\,, \qquad (6)$$

As the initial value the matrix which is obtained by the (4) or (5) is chosen. The criterion of orthogonality may be described by the following expression

$$\eta = \left\| A \cdot \hat{A}^{\mathrm{T}} - I_3 \right\|_F^2 ,$$

where $\| \, \|_F$ is the Frobenius norm (or Euclidian norm) of the matrix.

Least-squares criterion using for the matrix solution

Consider the solution of the attitude determination problem in the matrix form.

If we denote

$$\vec{\Delta}_i = \vec{b}_i - A\vec{r}_i ,$$

then expression (1) can be rewritten in the following form

$$L(A) = \frac{1}{2}\sum_{i=1}^{n} c_i \left\| \vec{\Delta}_i \right\|^2 = \frac{1}{2}\mathrm{tr}\left[(M - AM_0)C(M - AM_0)^{\mathrm{T}} \right],$$

where $C = \mathrm{diag}(c_1, c_2, \ldots, c_n)$.

The loss functional can be converted in such way:

$$L(A) = \frac{1}{2}\mathrm{tr}\left[(M - AM_0)C(M - AM_0)^{\mathrm{T}} \right] = \frac{1}{2}\mathrm{tr}\left[MCM^{\mathrm{T}} + CM_0^{\mathrm{T}}M_0 - 2AM_0CM^{\mathrm{T}} \right].$$

The constraint

$$A^{\mathrm{T}}A = I, \tag{7}$$

is here taken into account.

A problem is equivalent to maximizing of the expression

$$L_1(A) = \mathrm{tr}(AB),$$

where $B = M_0CM^{\mathrm{T}}$.

Taking into account (7) we obtain the loss functional in the next form

$$L_2(A) = \mathrm{tr}(AB) - \mathrm{tr}\left(\frac{1}{2}\Lambda\left(A^{\mathrm{T}}A - I\right) \right), \tag{8}$$

where Λ is the matrix of Lagrange factors.

If we pass to the vectors $\vec{r}_i^{\,*} = \sqrt{c_i}\vec{r}_i, \vec{b}_i^{\,*} = \sqrt{c_i}\vec{b}_i$ then the matrix **B** for the case $n = 3$ becomes look like

$$\mathrm{B} = \left[\vec{r}_1^{\,*}\vec{b}_1^{\,*} + \vec{r}_2^{\,*}\vec{b}_2^{\,*} + \vec{r}_3^{\,*}\vec{b}_3^{\,*} \right],$$

i.e., entering a matrix **C** is not necessary and $\mathbf{B} = \tilde{\mathbf{M}}_0\tilde{\mathbf{M}}^{\mathrm{T}}$, where matrices $\tilde{\mathbf{M}}_0$ and $\tilde{\mathbf{M}}$ contain vectors $\vec{r}_i^{\,*}$ and $\vec{b}_i^{\,*}$ respectively. Farther we will use these vectors.

Equating the derivative $L_2(A)$ for A to the zero, we obtain:

$$B^{\mathrm{T}} - \frac{1}{2}\left(A\Lambda^{\mathrm{T}} + A\Lambda \right) = 0 . \tag{9}$$

It is supposed that the matrix Λ is symmetric. Then (9) acquires the form $B^{\mathrm{T}} - A\Lambda = 0$, whence if follows

$$A = B^{\mathrm{T}}\Lambda^{-1} . \tag{10}$$

From the condition (9) we can find that

$$\Lambda = \sqrt{BB^{\mathrm{T}}} .$$

Finally we obtain such expression for the matrix **A**

$$\mathbf{A} = S\left(\sqrt{S^{\mathrm{T}}S} \right)^{-1} , \tag{11}$$

where $S = \tilde{M}\tilde{M}_0^T$.

For the case when the measurement errors are absent, the expression (11) becomes

$$A = \tilde{M}\tilde{M}_0^{\mathrm{T}}H_1^{-1} , \tag{12}$$

where $H_1 = \tilde{M}_0\tilde{M}_0^{\mathrm{T}}$.

This expression coincides with the direct solution (4) of equation $\tilde{M} = A\tilde{M}_0$. Really, since the matrix $\tilde{\mathbf{M}}_0$ in general case is not square, to find a matrix **A** at first it is needed multiply the left and right sides of this equation by $\tilde{M}_0^T$

$$\tilde{M}\tilde{M}_0^{\mathrm{T}} = AH_1 .$$

If we have three mutually perpendicular reference vectors (as in the TRIAD algorithm [4]), then $H_1 = I$ and

$$A = \tilde{M}\tilde{M}_0^{\mathrm{T}} .$$

Determining of the orientation angles based on individual elements of the attitude matrix

In accordance with (5) individual rows of attitude matrix **A** can be written as follows

$$\vec{p}_i = \left(M_0M_0^{\mathrm{T}} \right)^{-1} M_0\vec{q}_i ,$$

where $\vec{q}_1^{\,*} = \left[\vec{b}_{1x}^{\,*}, \vec{b}_{2x}^{\,*}, \ldots, \vec{b}_{nx}^{\,*} \right]^{\mathrm{T}}$; $\vec{q}_2^{\,*} = \left[\vec{b}_{1y}^{\,*}, \vec{b}_{2y}^{\,*}, \ldots, \vec{b}_{ny}^{\,*} \right]^{\mathrm{T}}$; $\vec{q}_3^{\,*} = \left[\vec{b}_{1z}^{\,*}, \vec{b}_{2z}^{\,*}, \ldots, \vec{b}_{nz}^{\,*} \right]^{\mathrm{T}}$ are column vectors which correspond to individual rows of matrix **M**, and $\vec{p}_1 = \left[a_{11}, a_{12}, a_{13} \right]^{\mathrm{T}}$, $\vec{p}_2 = \left[a_{21}, a_{22}, a_{23} \right]^{\mathrm{T}}$, $\vec{p}_3 = \left[a_{31}, a_{32}, a_{33} \right]^{\mathrm{T}}$ are column vectors which correspond to individual rows of matrix **A**,* defines disturbed vectors.

To determine rotation angles only five elements of attitude matrix are necessary. In fact three elements is sufficiently but the problem of the angles determining is much more complicated in this case. We use such sequence of rotation 3-2-1 (yaw angle ψ the first, pitch angle ϑ the second, roll angle φ the third). For this sequence of rotations elements of the first row and the third column of attitude matrix are necessary to calculate rotation angles. Then

$$\psi = \text{arctg}\left(\frac{a_{12}}{a_{11}}\right);$$

$$\vartheta = -\arcsin(a_{13}); \qquad (13)$$

$$\varphi = \text{arctg}\left(\frac{a_{23}}{a_{33}}\right).$$

As can be seen from (13) to determine rotation angles for our sequence of rotation it is necessary to have whole vector $\vec{p}_1$ and the third elements of $\vec{p}_2$ and $\vec{p}_3$. The last two elements we obtain from such expressions

$$a_{23} = H^{(3)}\vec{q}_2 \ a_{33} = H^{(3)}\vec{q}_3,$$

where

$$H = \left(M_0 M_0^{\mathrm{T}}\right)^{-1} M_0$$

where symbols (3) denote the third rows of matrix $\mathbf{A}$. In the case when rotation angles are small equations (13) can be simplified. Let us write expressions for required elements in detailed view

$$a_{11} = h_{11}b_{1x} + h_{12}b_{2x} + \ldots + h_{1n}b_{nx};$$
$$a_{12} = h_{21}b_{1x} + h_{22}b_{2x} + \ldots + h_{2n}b_{nx};$$
$$a_{13} = h_{31}b_{1x} + h_{32}b_{2x} + \ldots + h_{3n}b_{nx};$$
$$a_{23} = h_{31}b_{1y} + h_{32}b_{2y} + \ldots + h_{3n}b_{ny};$$
$$a_{33} = h_{31}b_{1z} + h_{32}b_{2z} + \ldots + h_{3n}b_{nz}.$$

From (8) – (12) we can conclude in ideal case yaw and pitch angles are defined only by *x*-projections of base vectors in body frame, while the roll angle is defined by the *y*- and *z*-projections of the vectors. But in real case where measurement errors are present each element depends on all three projections of base vectors.

Matrix **H** during some period of time can be considered as immutable. Elements of this matrix entirely depend on projections of base vectors in reference frame. In [9] it is shown that errors of base vectors in reference frame have small influence on attitude determination accuracy. Thereby such errors can be neglected. Immutability of matrix **H** allows reducing an amount of calculation needed to estimate a rotation angles.

Numerical simulation

In the absence of measurement errors, all attitude determination methods give the same result. However, these methods have different sensitivity to measurement errors. Therefore, an important problem is the impact of these errors on the accuracy of attitude determination. In this work vectors of measurement errors were modeled in accordance to the QUEST measurement model [4], [6]. Thus the measurement vectors were computed by means of the relationship

$$\vec{b}_i' = A\vec{r}_i + \Delta\vec{b}_i, \qquad (26)$$

where $\Delta\vec{b}_i$ is a vector of measurement noise which is assumed to be a zero-mean Gaussian and uniformly distributed in phase α about true vector with variance σ_i^2 per axis. Quantities σ_i are measured in radians. It is supposed that three vectors $\vec{b}_1 = [1,0,0]^T$, $\vec{b}_2 = [0,1,0]^T$, $\vec{b}_3 = [0,0,1]^T$ are available and the first and the third vectors are measured more precisely than the second one. Therefore, such values for σ_i were chosen $\sigma_1 = 0.0017$, $\sigma_2 = 0.0175$, $\sigma_3 = 0.0009$ rad. It was also intended that the body frame coincides with reference frame. Relative positions of direction vectors $\vec{b}_i$ and initial directions of counting angles α_i are shown on the Fig. 1.

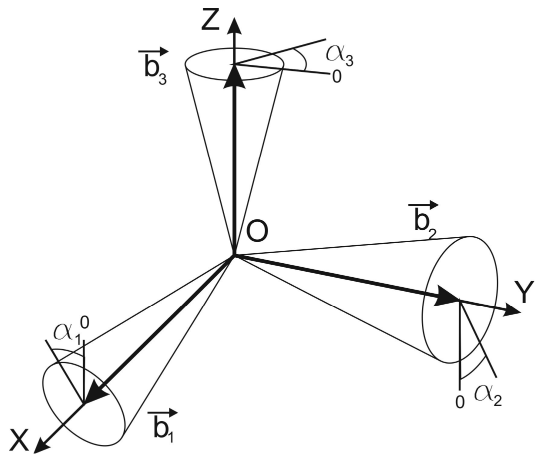

Fig. 1. Relative positions of direction vectors $\vec{b}_i$ and initial directions of counting angles α_i for deterministic case

The estimation error is measured by using the attitude error, as the principal angle of the corrective attitude matrix moving from the true to the estimated attitudes

$$\varepsilon = \arccos\left(\frac{1}{2}\left(\text{tr}\left(\hat{A}^{\mathrm{T}}A\right) - 1\right)\right).$$

The simulation was performed for methods described by (5) and (11). The procedure of ortogonalization (6) was performed for the first method. For comparison the TRIAD and SVD [3] methods were discussed. These methods are denoted by A5 (five elements), SR (square root), Tsb (TRIAD, Sun-Mag), SVD respectively. The results for the deterministic case (values σ_i and α are not generated but manually set) are represented in Figs 2, 3 and 4. The graphs for the values $\sigma_1 = 0.002$, $\sigma_2 = 0.02$, $\sigma_3 = 0.002$ rad respectively. Table contains the results for the random case (values σ_i and α are not generated). Number of tests N is equal to 1000.

As can be seen the attitude estimation errors of the method A5 do not depend on the phase α of the error vector. Although the A5 method use three measurement vectors it gives the worst results in comparison with the SR and SVD methods. This situation is caused by the fact that the weight coefficients (2) actually are not taken into account. This degrades the accuracy of the estimates.

The estimation results for diffenent methods for the random errors case ($N = 1000$)

Value	Method			
	TRIAD, sb	SR	A5	SVD
mean, deg	0.518	0.062	0.404	0.062
max, deg	3.021	0.254	1.954	0.254
std.dev., deg	0.466	0.044	0.299	0.044

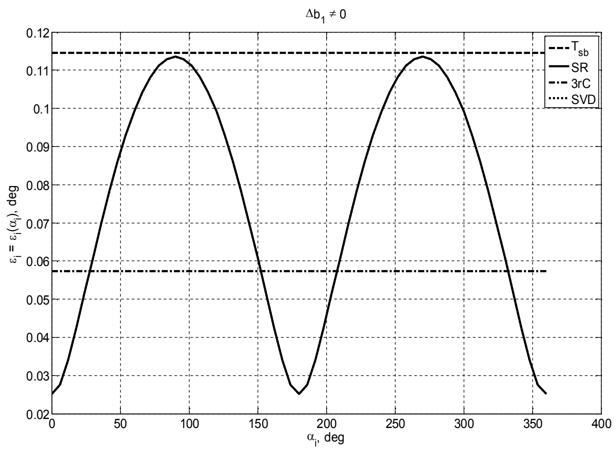

Fig. 2. Attitude estimation errors for the determinictic case s1 = 0.002 rad

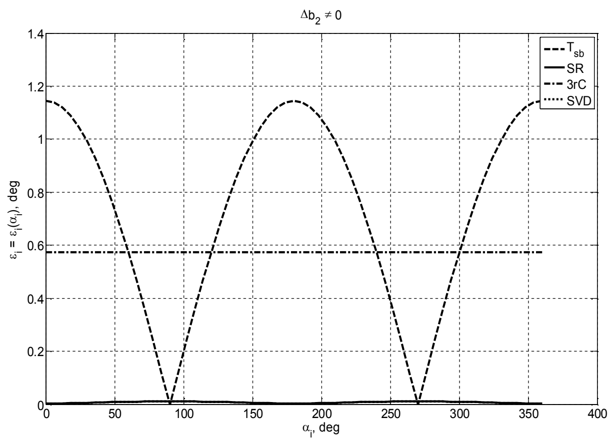

Fig. 3. Attitude estimation errors for the determinictic case s2 = 0.02 rad

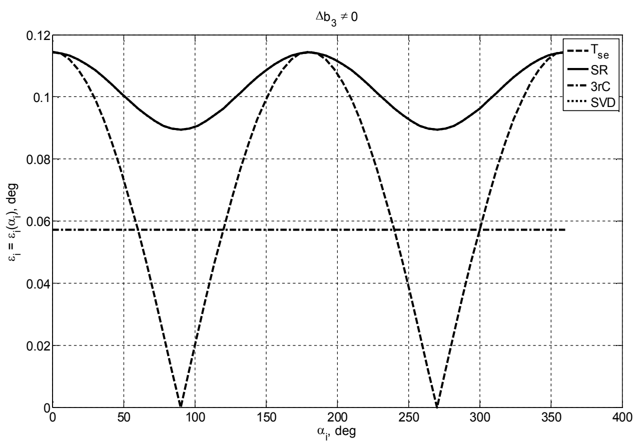

Fig. 4. Attitude estimation errors for the determinictic case s3 = 0.002 rad

Conclusion

This paper suggests the attitude estimation method which determines some elements of attitude matrix. This method can be considered as an analogue of the TRIAD algorithm for the case when more than two vector measurements are available. Since all vectors are treated equally then less accurate sensor will degrade the final estimate. If all the sensors are quite accurate the method will provide better results.

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Л. М. Рижков, Д. І. Степуренко. Визначення окремих елементів матриці орієнтації

Запропоновано метод визначення окремих елементів матриці орієнтації. Також представлено матричну форму розв'язку задачі визначення орієнтації літального апарату на основі критерію найменших квадратів. З використанням моделі вимірювань QUEST розглянуто питання впливу випадкових похибок вхідних даних на точність оцінювання орієнтації.

Ключові слова: метод оцінювання орієнтації; матрична форма розв'язку; задача Вахба; критерій найменших квадратів; модель вимірювання.

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Л. М. Рыжков, Д. И. Степуренко. Определение отдельных элементов матрицы ориентации
Предложен метод определения отдельных элементов матрицы ориентации. Также представлена матричная форма решения задачи оценивания ориентации летательного аппарата на основе критерия наименьших квадратов. С использованием модели измерений QUEST рассмотрен вопрос влияния случайных погрешностей входных данных на точность оценивания ориентации.
Ключевые слова: метод оценивания ориентации; матричная форма решения, задача Вахба; критерий наименьших квадратов; модель измерения.

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