UDC 531.383;629.7.05 (045)

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APPLICATION OF EXCEPTIONAL NON-NORMALIZED QUATERNION'S, FIVE-DIMENSIONAL ROTATION VECTORS AND THEIR ALGEBRAS IN PROBLEMS OF INERTIAL ORIENTATION

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Abstract. *Exceptional non-normalized quaternions, five-dimensional rotation vectors (pentanions) and their algebras in problems of inertial orientation are considered in this article.*

Keywords: exeptional non-normalised quaternion; pentanions; inertial orientaton.

Introduction

We are considering two types of new non-normalized exceptional rotation quaternion's of the rigid body $U = u_0 + \bar{\lambda}$, $V = v_0 + \bar{\lambda}$, where $u_0 = 1 - \lambda_0$, $v_0 = 1 + \lambda_0$; $\lambda_0 = \cos(\varphi/2)$, $\bar{\lambda} = \lambda \bar{k}$, $\lambda = \sin(\varphi/2)$; $\bar{k}$ – is the unit vector of Euler's axis of finite rotation (turn) of the solid in three-dimensional Euclidean vector space [1–3], φ is the rotation angle.

Parameter λ_0 and coordinates $\lambda_n (n = 1,\ 2,\ 3)$ of three-dimensional vector $\bar{\lambda}$ (coordinate orthonormal basis with unit vectors $\bar{i}_1$, $\bar{i}_2$, $\bar{i}_3$) related to a solid) are Euler [2] or Rodriguez–Hamilton [1–4] parameters. They define the classic Hamiltonian quaternion of rotation [1; 3] $\Lambda = \lambda_0 + \bar{\lambda}$ with unit norm $\left(\|\Lambda\| = \lambda_0^2 + \lambda^2 = 1,\ \lambda^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2\right)$.

Modified quaternions $U\left(\frac{1}{2}U\right)$ are examined in [5].

Unlike quaternion's Λ, non-normalized quaternion's *U*, *V* can be zero (at φ = 0 and φ = 2π respectively) and their modules depend on angle φ. Therefore, they are of special practical interest in solving two major problems [1; 2]: inertial sensing and inertial attitude control of the solid provided that the shortest turns (at angles φ < π and φ > π) are ensured. "Complete" definitely positive Lyapunov quaternion functions defined by all four parameters of quaternion's are relevant to use in solving the attitude control problems based on any rotation quaternion's. The mentioned parameters should (i) be independent variables of Lyapunov functions and (ii) go to zero in stable equilibrium positions of the solid (at φ = 0 and φ = 2π) as known from [6]. The above conditions are not met by classic Rodriguez–Hamilton parameters; however, they are met if non-normalized quaternions *U*, *V* parameters are used.

Problem statement

Quaternion's *U*, *V* form an exception (from the set of possible non-normalized quaternion's of rotation [1; 4; 5; 7; 8]) as those quaternion's and their corresponding kinematic differential equations, group multiplication formulas and quaternion rotation algebras have a number of special or unique properties. By way of example, quaternion's *U*, *V*, in addition to going to zero, have a common vector $\bar{\lambda}$ of quaternion Λ and their norms $\|U\|$, $\|V\|$ are equal to doubled scalar parts:

$$\|U\| = 2u_0 = U + \tilde{U} = U \circ \tilde{U} = u_0^2 + \lambda^2;$$

$$\|V\| = 2v_0 = V + \tilde{V} = V \circ \tilde{V} = v_0^2 + \lambda^2,$$

where $\tilde{U} = u_0 - \bar{\lambda}$, $\tilde{V} = \left(v_0 - \bar{\lambda}\right)$ are conjugate quaternion's, (∘) signifies quaternion (Hamiltonian) multiplication. In addition, the following equalities hold $u_0 + v_0 = 2$, $u_0 v_0 = \lambda^2 = \left(\bar{\lambda} \cdot \bar{\lambda}\right)$ (scalar product).

Kinematic differential equations for "Eigen" [1; 2] quaternion's *U*, *V*, are linear, but not homogeneous (unlike non-linear equations of other numerous non-normalized quaternion's [7] and homogeneous linear equation for quaternion Λ [1]). Those equations are obtained from kinematic equation for quaternion Λ [1] by substitution of variable λ_0 with variables u_0, v_0 and are as follows

$$2\dot{U} = 2\frac{du}{dt} = \Omega - \Omega \circ U, \quad 2\dot{V} = 2\frac{dv}{dt} = -\Omega + V \circ \Omega,$$

where $\Omega = \left(0 + \bar{\omega}\right)$ is the angular velocity quaternion, $\bar{\omega}$ is the vector of absolute rotational velocity of the solid.

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In order to integrate kinematic equations corresponding to quaternion's *U, V,* special high-precision computational algorithms, which transform quasi-coordinates (signals of laser, fiber-optic gyroscopes, gyroscopes based on elastic waves in solids, and Coriolis vibrating gyroscopes) into parameters of quaternion's *U, V,* may be synthesized in the onboard computers of strapdown inertial orientation systems (SIOS) and navigation systems (SINS) [2] (using the method described in [2]).

In performing it, numerous known [1; 2] high precision computational algorithms can also be used for Rodriguez-Hamilton parameters.

The multiplication formulas (rules) of [1; 2] non-normalized eigenquaternions *U, V,* are obtained from classic (group [7]) multiplication formulas of normalized eigenquaternions Λ by substitution of quaternion Λ with quaternion's *U, V,* according to the following formulas $\Lambda = E_4 - \tilde{U} = V + E_4$, where $E_4 = \left(1 + \bar{0}\right)$ is a unit quaternion. The group multiplication formulas of normalized quaternions Λ and non-normalized quaternion's *U, V* for 2 sequential finite rotations (turns) of the solid are written as

$$\Lambda = \Lambda' \circ \Lambda'', \quad U = U' \otimes U'' = U' + U'' - U'' \circ U';$$

$$V = V' \otimes V'' = 2E_4 - V' - V'' + V' \circ V'',$$

where Λ, *U, V* are the resulting rotation quaternions; Λ', *U'*, *V'* are the first rotation quaternions; Λ'', *U''*, *V''* are the second rotation quaternions; ($\otimes$) signifies group (non-Hamiltonian) multiplication [7] of non-normalized quaternions.

The following equalities follow from the above formulas $U \otimes \tilde{U} = \tilde{U} \otimes U = 0$ and $V \otimes \tilde{V} = \tilde{V} \otimes V = 2E_4$, where $0 = \left(o + \bar{o}\right)$ are zero quaternions and $\bar{o}$ is a zero vector.

The quaternion sets Λ, *U, V,* form groups [9] of rotation quaternion's of the solid or quaternion groups of three-dimensional rotations with the above group multiplication formulas. Those formulas (as algebraic equations) have a unique solution for factors Λ', *U'*, *V'* or Λ'', *U''*, *V''* according to the following quaternion division formulas $\Lambda' = \Lambda \circ \tilde{\Lambda}''$ and $\Lambda'' = \tilde{\Lambda}' \circ \Lambda$; $U' = U + \tilde{U}'' - \tilde{U}'' \circ U$ and $U'' = U + \tilde{U}' - \tilde{U}' \circ U$; $V' = 2E_4 - \tilde{V}'' + V \circ \tilde{V}''$ and $V'' = 2E_4 - V - V' + V \circ V'$.

In scalar-vector notation [1], the multiplication formulas of quaternion's *U, V,* are used in order to build (in the Cauchy's form [2]) general solutions of the kinematic differential equations. Such solutions are required in order to arrive at recursive formulas in the computational algorithms of parameters for quaternion's *U, V* in SIOS or SINS [2].

The spaces of non-normalized quaternion's *U, V* in conjunction with their multiplication formulas define new real, associative, non-commutative and non-normalized quaternion rotation algebras. Those algebras have division [10] with zero divisors

$$U' = \tilde{U}'', \; U'' = \tilde{U}'; \; V' = 2E - \tilde{V}'', \; V'' = 2E_4 - \tilde{V}'$$

($u_0' = u'' = 1$, $v_0' = v_0'' = 1$ and $\bar{\lambda}' = \bar{\lambda}''$ at $\varphi' = \varphi'' = \pi$, $\lambda_0 = 0$). They differ significantly from the algebras of other non-normalized quaternion's of rotations [7; 8] having considerably more complicated group multiplication formulas [7].

The parameters of quaternion's *U, V* are used in definitely positive Lyapunov quaternion functions f_U, f_V which are quadratic [8]:

$$f_U = \alpha_U u_0^2 + \beta_U \left(\bar{\lambda} \cdot A_U \bar{\lambda}\right) + \gamma_U \left(\bar{\omega} \cdot \bar{g}\right);$$

$$f_V = \alpha_V v_0^2 + \beta_V \left(\bar{\lambda} \cdot A_V \bar{\lambda}\right) + \gamma_V \left(\bar{\omega} \cdot \bar{g}\right),$$

where $\alpha_U, \beta_U, \gamma_U > 0$ and $\alpha_V, \beta_V, \gamma_V > 0$; A_U, A_V are definitely positive symmetric constant operators; $\bar{g} = J\bar{\omega}$ is the momentum of the solid; J is the inertia operator (tensor) of the solid; $\bar{\omega}$ is the angular velocity vector of the solid. Function f_U at $u_0 < 1$, $v_0 > 1$ $(0 < \varphi < \pi)$ or function f_V at $u_0 > 1$, $v_0 < 1$ $(\pi < \varphi < 2\pi)$ are used to provide control of the shortest turns of the solid.

The appropriate selection of the formulas to determine the control moment vectors (using the method described in [7]) provides negative definiteness of derivatives of $\dot{f}_U$, $\dot{f}_V$ of Lyapunov functions over time. This results in asymptotic stability of attitude control processes of the solid and its shortest turns in the entire range of angle φ from 0° to 360°. Thus defines new quaternion-based laws and algorithms for the inertial attitude control of the solid different from the known laws and algorithms, which use Rodriguez–Hamilton parameters (see, by way of example, [11]) and defining no space of variables, in which the Lyapunov functions are represented.

Based on normalized rotation quaternion's Λ and non-normalized rotation quaternion's *U, V* we obtain five-dimensional rotation vectors in the form of $\bar{p} = p_0 \bar{i}_0 + \bar{\lambda} + p_4 \bar{i}_4$, where $\bar{\lambda}$ is a three-dimensional vector $\bar{\lambda} = \lambda_1 \bar{i}_1 + \lambda_2 \bar{i}_2 + \lambda_3 \bar{i}_3$ in quaternion's Λ, *U, V*; p_0, p_4 are any two of the three scalar parameters u_0, v_0, λ_0, $\bar{i}_0$ ... $\bar{i}_4$ are unit vectors of a certain five-dimensional normalized orthogonal coordinate

basis; $p_0, \lambda_1, \lambda_2, \lambda_3, p_4$ are the coordinates $p_m (m = 0, 1, 2, 3, 4)$ of vector $\bar{p}$ in that basis.

Vectors $\bar{p}$ are elements in five-dimensional Euclidean vector space over the real number field [3]. That space turns [3] to five-dimensional vector rotation algebra by defining for it a group multiplication operation (*) of 2 five-dimensional arbitrary vectors $\bar{p}', \bar{p}''$, which assigns to them the third vector $\bar{p}$, i. e. operation $\bar{p} = \bar{p}' * \bar{p}''$.

The multiplication (and division) operations in five-dimensional rotation algebras are defined by the three combinations of the group multiplication (and division) formulas for parameters of quaternion's Λ, *U*, *V*.

This results in real, associative, and non-commutative five-dimensional vector rotation algebras with unambiguous division and without zero divisors [10] (as vectors $\bar{p}$ cannot be zero).

In order to solve attitude control problems of the solid we use five-dimensional vector $\bar{p}_{UV} = u_0 \bar{i}_0 + \bar{\lambda} + v_0 \bar{i}_4$ with multiplication formulas

$$u_0 = u_0' + u_0'' - \left(u_0' u_0'' - \left(\bar{\lambda}' \bar{\lambda}''\right)\right),$$

$$\lambda = \frac{1}{2}\left(v_0'' - u_0''\right)\bar{\lambda}' + \frac{1}{2}\left(v_0' - u_0'\right)\bar{\lambda}'' + \bar{\lambda}' \times \bar{\lambda}'',$$

$$v_0 = 2 - v_0' - v_0'' + v_0' v_0'' - \left(\bar{\lambda} \cdot \bar{\lambda}_0''\right).$$

Here, $\frac{1}{2}\left(v_0'' - u_0''\right) = \lambda_0''$, $\frac{1}{2}\left(v_0' - u_0'\right) = \lambda'$, $\bar{\lambda}' \times \bar{\lambda}''$ – is a vector product.

The system of kinematic differential equations for parameters u_0, λ_n, v_0 is linear and is written as follows $2\dot{u}_0 = \left(\bar{\omega} \cdot \bar{\lambda}\right)$, $2\dot{\bar{\lambda}} = \frac{1}{2}\left(v_0 - u_0\right)\bar{\omega} + \bar{\lambda} \times \bar{\omega}$; $2\dot{v}_0 = -\left(\bar{\omega} \cdot \bar{\lambda}\right)$ in the scalar-vector notation.

When defining the control moment vectors we use definitely positive general Lyapunov function f_{UV} as follows [8]:

$$f_{UV} = \alpha_{UV} u_0^2 + \beta_{UV}\left(\bar{\lambda} \cdot A\bar{\lambda}\right) + \gamma_{UV} v_0^2 + \delta\left(\bar{\omega} \cdot \bar{g}\right),$$

where $\delta > 0$; $\beta_{UV} > 0$; $\alpha_{UV} > 0$; $\gamma_{UV} = 0$ at $0 < u_0 < 1$; $1 < v_0 < 2$; $\alpha_{UV} = 0$; $\gamma_{UV} > 0$ at $1 < u_0 < 2$; $0 < v_0 < 1$. Here, operator *A* is a definitely positive symmetric constant operator.

The control moment vector is defined based on the requirement to ensure asymptotic stability of the attitude control process of the solid at $f_{UV} < 0$ [7].

Five-dimensional rotation vectors $\bar{p}$ produce (by analogy to quaternion's) hypercomplex [12; 13] five-dimensional systems: *pentanions* of rotation of the solid. The pentanions are written (without unit vector $\bar{i}_0$ [12]) as $P = p_0 + \lambda_1 \bar{i}_1 + \lambda_2 \bar{i}_2 + \lambda_3 \bar{i}_3 + p_4 \bar{i}_4 = p_0 + \bar{\lambda} + p_4 \bar{i}_4$, where p_0 is the scalar part of the pentanion, $\left(\bar{\lambda} + p_4 \bar{i}_4\right)$ is its vector part, P_m are the parameters of the pentanion. Norm $\|P\|$ of the pentanion is defined as a scalar product: $\|P\| = \left(\bar{p} \cdot \bar{p}\right) = p_0^2 + \left(\bar{\lambda} \cdot \bar{\lambda}\right) + p_4^2$. The pentanion algebras of rotation are isomorphic to the corresponding algebras of five-dimensional rotation vectors and all formulas for five-dimensional vector algebras are simply rewritten to the corresponding formulas of the pentanion algebras of rotation.

The five-dimensional algebras, by way of example, of pentanions $P_{\Lambda U} = \lambda_0 + \bar{\lambda} + u_0 \bar{i}_4$, $P_{\Lambda V} = \lambda_0 + \bar{\lambda} + v_0 \bar{i}_4$, contain classic normalized quaternion algebra Λ, which, in turn, contains the algebra of complex numbers and the algebra of real numbers. This results in a question on possibility to refine or newly generalize the Frobenius theorem [10; 12] (to the five-dimensional algebras), which states that the algebra of quaternion's is the unique finite dimensional real associative, but not commutative algebra without zero divisors.

We are considering a problem of approximate determination and correction computational error in SIOS caused by an algorithmic inaccuracy of SIOS and rounding-off errors in onboard computers of SIOS using the parameters of pentanions *P*. In order to determine the computation errors of parameters u_0, v_0 and module λ, we use exact and approximate equalities of the form. We are considering the possibility to determine the errors of Rodriguez–Hamilton parameters

$\|U\| = 2u_0$, $\|V\| = 2v_0$, $u_0 + v_0 = 2$, $u_0 v_0 = \lambda^2$ based on separate and joint computations of the parameters $\lambda_n (n = 1, 2, 3)$ and parameters $u_0 v_0$ by using equalities

$\lambda_U^2 = 2u_0 - u_0^2$, $\lambda_V^2 = 2v_0 - v_0^2$, $\bar{\lambda} = \lambda \bar{k}$, $\left(\bar{k} \cdot \bar{k}\right) = 1$, $\lambda^2 = 1 - \lambda_0^2$.

Conclusions

Based on the pentanions of rotation we provide five-dimensional parameterization of the three-dimensional rotation group. The properties of that parameterization significantly and favorably differ from the known five-dimensional Hopf parameterization [14].

Researched properties of pentanions can be used in algorithms of strapdoun inertial navigation systems.

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Received 26 September 2013

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А. П. Панов, В. В. Цисарж. Застосування виключних ненормалізованих кватерніонів, пятивимірних векторів обертання та їх алгебри у задачах інерціальної орієнтації

Розглянуто виключні ненормалізовані кватерніони, п’ятивимірні вектори обертання та їх алгебри у задачах інерціальної орієнтації.

Ключові слова: виключні ненормалізовані кватерніони; пентарніони; орієнтація.

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А. П. Панов. В. В. Цисарж. Применение исключительных ненормализированых кватернионов, пятимерных векторов вращения и их алгебры в задачах инерциальной ориентации

Рассмотрены исключительные ненормализированные кватернионы, пятимерные векторы вращения и их алгебра в задачах инерциальной ориентации.

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