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MINIMIZATION OF ANALOG-TO-DIGITAL SIGNAL CONVERSION QUANTIZATION NOISE

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Abstract. *An algorithm of signal processing during the analog-to-digital conversion phase is discussed and substantiated. This algorithm is aimed at minimization of quantization noise power. The results of simulation are given in order to confirm the effectiveness of the proposed algorithm.*

Keywords: digital-to-analog conversion; quantization levels; distribution function; quantization characteristics; signal; quantization noise; variance; minimization; efficiency.

Introduction and the Problem Statement

Commercially available analog-to-digital converters (ADCs) typically have constant quantization step throughout the dynamic range. Thus quantization noise power is defined only by the ADC resolution.

It is known [1; 2] that the quantization noise power can be reduced by taking into account the probabilistic characteristics of the converted signal. In this case quantization levels that refer to subintervals of dynamic range of higher probability density should be placed more frequently than quantization levels that refer to subintervals of lower probability density.

There is a plenty of sources containing recommendations on non-linear quantization level setting, e. g. [1] offers using “dynamic compression” and defining quantization level values with a logarithmic function. Some methods of companding are suggested in [3]. A- and μ-laws [2; 4; 5] are commonly known as they are based on some features of speech signals.

In fact, all the suggestions are based on the hypothesis claiming that in order to minimize the quantization noise a special non-linear signal conversion should be performed. The conversion should turn signal probability density into a uniform distribution. For a signal with a uniform distribution constant quantization step is the most appropriate. To calculate this conversion, simply reverse function to the distribution function of the original signal should be applied [6]. However, the issue of quantization noise power after signal recovery from digital to analog form still remains.

The results of the statistical simulation [7] of digital conversion with signal recovery to analog form afterwards displayed that applying non-linear conversion that turns signal probability density function into uniform one is not optimal as quantization noise power in this case is not minimized. Thereby, the problem is formulated as the following.

An analog-to-digital converter input receives analog signal, distribution law of which is known. Quantization characteristics type minimizing quantization noise power is to be found.

Synthesis of Optimal Quantization Characteristics

The optimality criterion is determined as criteria of minimum mean-square deviation σ of input signal samples from recovered after quantization signal samples:

$$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(U_i - V_i)^2},$$

where V is a vector of original (after sampling operation applied to analog signal) signal samples, U is a vector of quantized signal samples, N is a total quantity of samples.

Let’s define quantization characteristics $Q(p)$, it will determine the quantization level values as follows: the first level will be $Q(1/n)$, the second level $Q(2/n)$ etc., where n is a total number of quantization levels. Since in real systems discrete signal values are limited, let’s define minimum value as zero and maximum value as one.

To execute this, signal samples should be normalized by the dynamic range. Let X (in volts) be sample value before normalization, and x (dimensionless) – sample value after normalization. Let $[X_{\min}; X_{\max}]$ be the dynamic range. Then the value of x will be calculated by the following formula:

$$x = \frac{X - X_{\min}}{X_{\max} - X_{\min}}.$$

Here and further let’s assume that the values of the signal samples have already been normalized and are dimensionless quantities.

Thus the distribution function $F(x)$ will take the values of 0 and 1 at 0 and 1, respectively: $F(0)=0,\ F(1)=1$.

Solution of the problem reduces to finding such a quantization characteristic $Q(p)$ that provides minimization of the mean-square deviation of input signal samples values from recovered after quantization values, while distribution function $F(x)$ is given. To assess the efficiency of proposed algorithm, let's introduce gain ε, which shows how much the quantization noise power has reduced comparing to the uniform quantization:

$$\varepsilon = \frac{\sigma_{unif}^{\ 2} - \sigma_{nonunif}^{\ 2}}{\sigma_{unif}^{\ 2}},$$

where σ_{unif} is a mean-square deviation in case of uniform quantization, $\sigma_{nonunif}$ is a mean-square deviation in case of non-uniform quantization with found characteristics $Q(p)$.

Consideration of Elementary Particular Case

Let's consider particular case of non-uniform input signal sample distribution: a sample falls into the interval $(0;\,x_0)$ with a certain probability p_0, otherwise (with a probability $(1-p_0)$) it falls into $(x_0;\,1)$; samples have uniform distribution within each of these two intervals. Than distribution function will look like broken line as shown in fig. 1. Solution of the quantize function for this particular case allows to approach the general problem later.

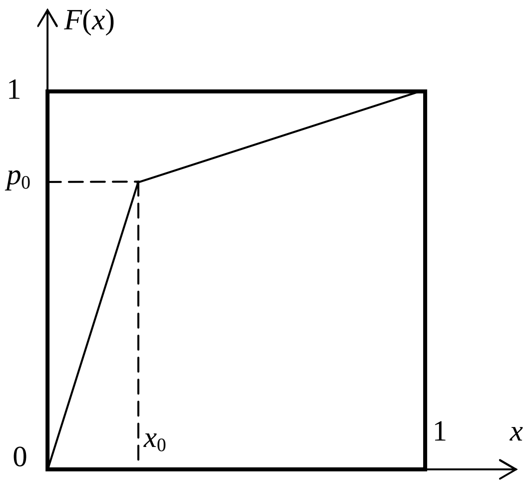


Fig. 1. Elementary case of non-uniform signal sample distribution. $F(x)$ is a distribution function of normalized signal samples

Let's prove the following statement.

Let the distribution function $F(x)$ consist of two linear segments (fig. 1) and have point of fracture (x_0; p_0). Quantization characteristics $Q(p)$ providing the least value of $\sigma_{nonunif}$ for the given signal sample distribution is to be found.

Unknown function $Q(p)$ will consist of two linear segments with the point of fracture (φ; x_0) (fig. 2). Parameter φ is defined from the equations:

$$\varphi = \frac{\sqrt[3]{c_1}}{\sqrt[3]{c_1} + \sqrt[3]{c_2}},$$

where $c_1 = p_0 x_0^{\ 2};\quad c_2 = (1-p_0)(1-x_0)^2.$ (1)

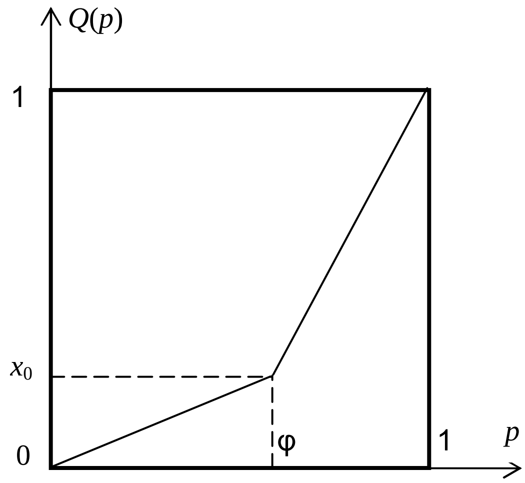


Fig. 2. Quantization characteristics which refers to the distribution function shown in Fig. 1

Proof. Let n be total number of quantization levels. Obviously, the greater is the slope of the distribution function in an interval, the more often quantization levels should be positioned in this interval. Thus the quantization characteristic will also consist of two segments, the ordinate of the fracture will be equal to x_0; quantization characteristic line itself will be convex in a way opposite to the convexity of $F(x)$ line. Abscissa of the point of fracture will be our unknown quantity.

Consider intervals $[0;x_0]$ and $[x_0;1]$ separately.

At the interval $[0;x_0]$ the following statements are valid:

– there are Np_0 signal samples within (N is considered to be sufficiently high);

– quantization interval width is $\Delta_{q1} = \dfrac{x_0}{\varphi n}$;

– quantization characteristics line is expressed by equation $Q(p) = \dfrac{x_0}{\varphi} p$.

At the interval $[x_0;1]$:

– there are $N(1-p_0)$ signal samples within;

– quantization interval width is $\Delta_{q2} = \dfrac{1 - x_0}{(1 - \varphi)n}$;

– quantization characteristics line is expressed by equation $Q(p) = \dfrac{(1 - x_0)(p - \varphi)}{1 - \varphi} + x_0$.

Let Δ_i be difference between sample values before and after quantization: for $i = \overline{1, N}$ $\Delta_i = |U_i - V_i|$. After sample highest ranking the first Np_0 will fall into the interval $[0; x_0]$, the rest – into the interval $[x_0; 1]$. We suppose that for every $i = \overline{1, N}$ $\Delta_i = 0{,}5\Delta_q$, i. e. each signal sample falls exactly in the middle between adjacent quantization levels (extreme case).

For the considered case the following is valid:

$$\sigma = \left(\frac{1}{N}\sum_{i=1}^{N}\Delta_i^2\right)^{1/2} = \frac{1}{\sqrt{N}}\left(\sum_{i=1}^{[Np_0]}\Delta_i^2 + \sum_{i=[Np_0]+1}^{N}\Delta_i^2\right)^{1/2} =$$

$$= \frac{1}{\sqrt{N}}\left(Np_0\left(0{,}5\frac{x_0}{\varphi n}\right)^2 + N(1 - p_0)\left(0.5\frac{(1 - x_0)}{(1 - \varphi)n}\right)^2\right)^{1/2} =$$

$$= \frac{0{,}5}{n}\sqrt{\frac{p_0 x_0^2}{\varphi^2} + \frac{(1 - p_0)(1 - x_0)^2}{(1 - \varphi)^2}}. \quad (2)$$

Applying previously defined variables $c_1 = p_0 x_0^2$; $c_2 = (1 - p_0)(1 - x_0)^2$ and normalizing mean-square deviation (2) by quantization level quantity ($\sigma_n = \sigma \cdot 2n$, $n = \text{const}$) we come to the following:

$$\sigma_n = \sqrt{\frac{c_1}{\varphi^2} + \frac{c_2}{(1 - \varphi)^2}}. \quad (3)$$

Let's differentiate σ_n with respect to φ:

$$\frac{d\sigma_n}{d\varphi} = \frac{-\dfrac{2c_1}{\varphi^3} + \dfrac{2c_2}{(1 - \varphi)^3}}{2\sqrt{\dfrac{c_1}{\varphi^2} + \dfrac{c_2}{(1 - \varphi)^2}}}, \quad (4)$$

and after equating (4) to zero, we find the critical point of the function (3):

$$\varphi = \frac{\sqrt[3]{c_1}}{\sqrt[3]{c_1} + \sqrt[3]{c_2}}.$$

It is easy to check that this point is a minimum point of $\sigma_n(\varphi)$, Q.E.D.

Solution of the Problem for the General Case

We will perform a piecewise linear approximation of the distribution function consistently with two, four, eight etc., at last 2^k linear segments, forming a piecewise quantization characteristics on every step.

Algorithm to obtain the quantization characteristics is as follows:

1) initially put $l_{0,0}$ a segment of the curve $F(x)$ within interval $[0;1]$, $q_{0,0}$ a line of quantization characteristics initially set as $Q(x) = x$ within interval $[0;1]$;

2) approximate piecewisely the curve segment $l_{i,k}$ of the distribution function with a two-segmented broken line. The point of fracture should lie on $F(x)$ curve. This point divides $l_{i,k}$ into segments $l_{2i,k+1}$ and $l_{2i+1,k+1}$;

3) replace the line $q_{i,k}$ of quantization characteristics with a two-segmented broken line with segments $q_{2i,k+1}$ and $q_{2i+1,k+1}$ using (1) (the segments of the distribution function and quantization characteristics must be renormalized relatively to the endpoints of segment $l_{i,k}$ and line $q_{i,k}$);

4) if $i = 2^k - 1$, go to step 2 setting parameter $i = 0$ and parameter k, incremented by one; otherwise do not change parameter k and increment i by one, then return to step 2.

Steps 2–4 could be repeated as many times as possible: the more is final value of k, the more accurately approximation of the distribution function is performed and the more is gain value.

The algorithm is illustrated in fig. 3 as an example of being applied to the Gaussian law (second, third iteration and the final result).

To quantify the efficiency of the proposed algorithm of quantization noise reduction, statistical simulation with various signal distribution laws has been performed. In particular, for the Gaussian distribution law $\pm 3\sigma$ value restrictions were implemented. To enclose the signal fully within the interval (0; 1) expectation value $M = 0{,}5$ and standard deviation $\sigma = 1/6$ were taken. In this case, the distribution function could be expressed as follows:

$$F(x) = \int_0^x \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(\xi - M)^2}{2\sigma^2}\right) d\xi,$$

where $M = 0.5$, $\sigma = 1/6$.

The number of input signal samples was set to $N = 8000$, the number of quantization levels – to $n = 4096$ (12-bit ADC).

During approximation the *App*(x) function consisting of 16 linear segments has been generated. The segments were formed as a result of a four-step successive approximation. As a quality criterion of approximation we used the area between the graph of $F(x)$ and a polygonal line:

$$\int_0^1 \left| F(x) - App(x) \right| \mathrm{d}x.$$

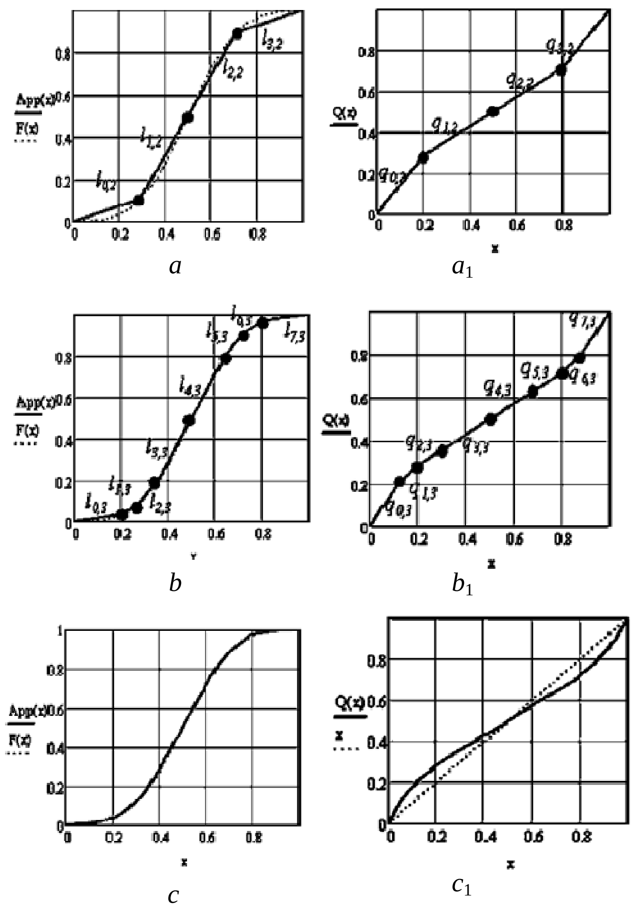


Fig. 3. Illustration of the algorithm implementation to normal distribution: *a*, a_1 – approximation function and quantization characteristics after the second iteration; *b*, b_1 – after the third iteration; *c*, c_1 – final result

The results of statistical simulation revealed that for Gaussian input signal, the gain is ε = 27 %, when the suggested algorithm is applied, i.e. the quantization noise power has been decreased by 27 %, though ADC bit resolution remain the same.

Fig. 4 shows a graph of the quantization characteristic (solid line) and the distribution function (dotted line) for this case.

Quantization functions were also obtained using similar method for such distribution functions:

– exponential;

– Rayleigh.

Let $L(x)$ be an inverse function to quantization characteristics $Q(p)$ and let's render it on the graphs next to the distribution functions for each of the distribution laws above.

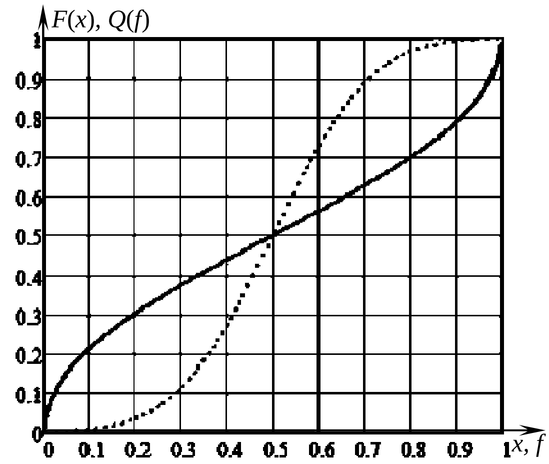


Fig. 4. Normal distribution (dotted line) and the corresponding quantization characteristic (solid line)

Having looked at these graphs, we can notice an obvious regularity: the function $L(x)$ takes some "intermediate position" between the function $F(x)$ and the function $p = x$ (fig. 5). Therefore we decided to look for a function in the form like below:

$$L(x) = (F(x) - x)\beta + x, \qquad (5)$$

where β is a coefficient lying in range (0; 1) depending on the function $F(x)$.

Let's prove a statement: **for quantization characteristics being the inverse functions to *P*(*x*) (*P*(*x*) is calculated by formula (5)), on condition of quantization levels number and discrete samples number tending to infinity, the quantization noise power is proportional to:**

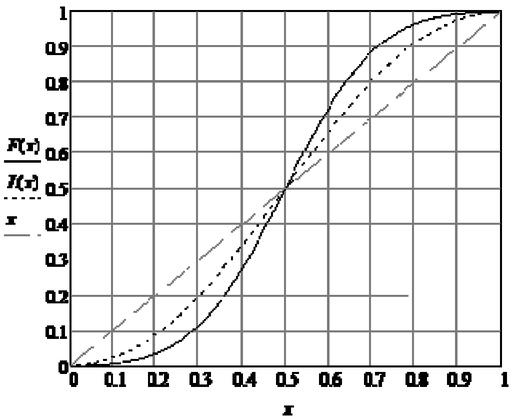
$$\Delta_n = \int_0^1 \frac{F'(x)dx}{[\beta F'(x) - \beta + 1]^2}.$$

Proof. Let $x_1, x_2, ..., x_n$ be the quantization level values. Let's consider *i*th interval of dynamic range (which is set to $[0;1]$, like before), i.e. $[x_i; x_{i+1}]$. For sufficiently large *n*, we consider that this interval contains the number of samples proportional to the value of $F'(x_i)$ (derivative with respect to *x*). Quantization error in this interval is proportional to $\Delta_q = (x_{i+1} - x_i)n$. As a consequence of quantization charactestic definition $Q\left(\frac{i}{n}\right) = x_i$, hence $L(x_i) = \frac{i}{n}$.

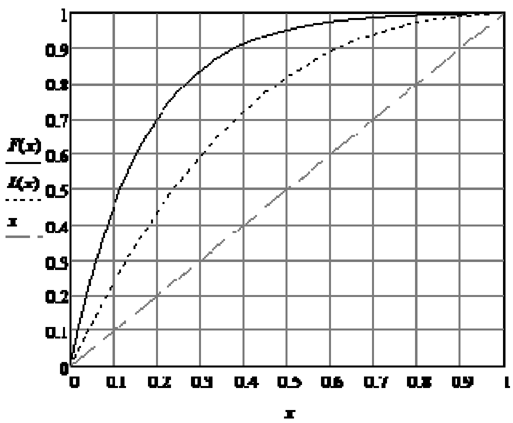
It follows:

$$\Delta_q = (x_{i+1} - x_i)n = \frac{x_{i+1} - x_i}{\frac{i+1}{n} - \frac{i}{n}} = \frac{x_{i+1} - x_i}{L(x_{i+1}) - L(x_i)}; \qquad (6)$$

$$\lim_{n\to\infty} \Delta_q = \frac{1}{L'(x_i)}.$$



a



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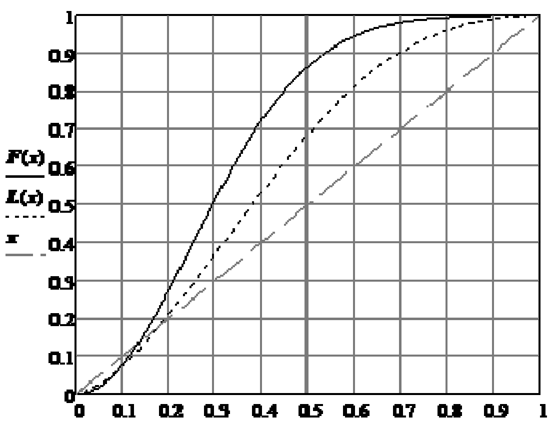

c

Fig. 5. Illustration of the algorithm implementation for the following distribution laws: Gaussian (*a*), exponential (*b*), Rayleigh (*c*). Solid line is the distribution function, dashed is the inverse function $L(x)$ of the quantization characteristic, dashed line is a function $p = x$

Summing over all intervals and substituting the formula (5), on condition of $n \to \infty$ we obtain:

$$\Delta_n = \int_0^1 \frac{F'(x)dx}{[\beta F'(x) - \beta + 1]^2}, \tag{7}$$

quod erat demonstrandum.

Note: As discussed earlier in the particular case, when using uniform quantization $\Delta_n = 1$.

Next task is to find the coefficient β for which the function (7) reaches its minimum. This is equivalent to finding a zero of the function (8):

$$\frac{d\Delta_n}{d\beta} = \int_0^1 \frac{F'(x)(F'(x)-1)dx}{[\beta F'(x) - \beta + 1]^3} = 0. \tag{8}$$

The example of $\Delta_n(\beta)$ relation for exponential distribution function is shown on fig. 6.

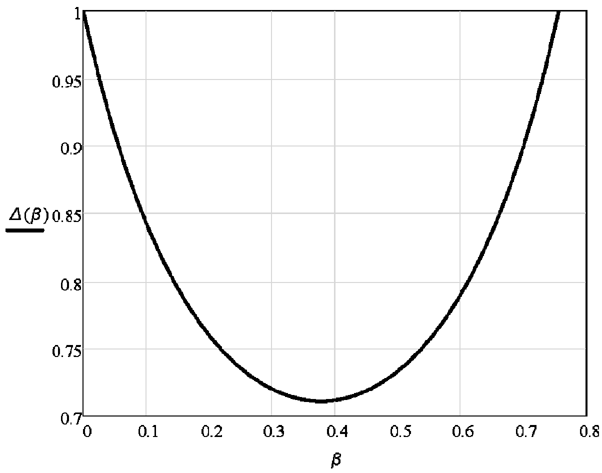

Fig. 6. Graph of relation $\Delta_n(\beta)$

A statistical simulation of building the quantization characteristics on the basis of satisfying the condition (8) coefficient β has been performed.

As a result, the following gain ε values have been obtained:

– 27 % for the Gaussian distribution (expectation value M = 0,5, standard deviation σ = 1/6);

– 29 % for the exponential distribution (rate parameter $\lambda = 4$).

Conclusion

The proposed method of analog-to-digital conversion optimization allows the simplest means to reduce the level of quantization noise without increasing the ADC bit resolution.

Priori uncertainty related to absence of complete probabilistic characteristics of the converted signal can be overcome by using adaptive modifications of the proposed algorithm.

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О. І. Давлет’янц, І. В. Харламов. Мінімізація шуму квантування при аналого-цифровому перетворенні

Запропоновано й обґрунтовано методику й алгоритм обробки сигналів на етапі аналого-цифрового перетворення з метою забезпечення мінімуму потужності шуму квантування. Приведені результати статистичного моделювання, які ілюструють ефективність пропонованого алгоритму.

Ключові слова: сигнал; аналого-цифрове перетворення; розміщення порогів; функція розподілу; характеристика квантування; шум квантування; дисперсія; мінімізація; ефективність**.**

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А. И. Давлетьянц, И. В. Харламов. Минимизация уровня шума квантования при аналого-цифровом преобразовании сигнала

Предложены и обоснованы методика и алгоритм обработки сигналов на этапе аналого-цифрового преобразования с целью обеспечения минимума мощности шума квантования. Приведены результаты статистического моделирования, иллюстрирующие эффективность предлагаемого алгоритма.

Ключевые слова: сигнал; аналого-цифровое преобразование; расстановка порогов; функция распределения; характеристика квантования; шум квантования; дисперсия; минимизация; эффективность.

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