OPTIMAL PRINCIPLE FOR DYNAMICAL SYSTEM WITH ALTERNATIVE ORBITING

1Department of Telecommunications, National Technical University of Ukraine
“Igor Sikorsky Kyiv Polytechnic Institute”, Kyiv, Ukraine
2Department of Automation and Energy Management, National Aviation University, Kyiv, Ukraine
E-mails: 1lysenko.a.i.1952@gmail.com, 2tachinina@rambler.ru

Abstract—Lagrange problem with the account of functional limitation at any functional limitations at any moment in a given interval is presented. The required conditions for optimal trajectories of the determined dynamic system synthesized as space vehicle trajectories have been obtained.

Index Terms—Optimization; Dynamic system; Branching path; Optimal trajectory.

I. INTRODUCTION

At present the problem is considered when the dynamic system trajectories should satisfy not only the main trajectories but a number of alternative ones [1].

Trajectory alternativity consists in that at any moment of dynamic system movement on this trajectory there are conditions for another variant of movement the objective of which excludes the main objective of this system movement. To such class of trajectories one may relate for example flying vehicle landing trajectory from any point of which go-around fly is possible; trajectories of flying vehicle orbital injection enabling the maneuver of return in the event of malfunction [2], vehicle flight trajectory from any point of given section of which accident-free cargo dropping is possible [3]. The designed possibility of dynamic system trajectory branching at any moment in a given interval gives grounds to refer to such a trajectory as quasi-branching trajectory or trajectory with an alternative (alternative variant of movement).

II. PROBLEM STATEMENT

The proposed in [1] Boltz problem formulation contemplates dynamic system movement main trajectory optimization provided that redirecting and (or) system movement dynamics change requiring the construction of auxiliary trajectory and resulting in main trajectory branching may occur at a finite number of points of the main trajectory. But in a whole number of technical systems the command of change to auxiliary trajectory may come at any current moment in a given interval [2]. This means that it is necessary to solve branched trajectory optimization problem with not fixed time moment but with fixed time interval that is with infinite number of fixed points of main trajectory branching.

The derivation of the required conditions of optimality for main trajectory movement of the dynam-
The problem becomes considerably more complicated for the case of disintegrated system with current moment of disintegration.

We consider the problem of optimization for disintegrated system

\[
I = S(x(t_0), t_0); x(t_1), t_1; x(t_2), t_2; \ldots; x(t_N), t_N; x(t_f), t_f)
\]

\[
+ \int_{t_0}^{t_f} \Phi(x, u, t) dt + \int_{t_0}^{t_f} \Phi^0(x^0, u^0, \eta) d\eta \rightarrow \min,
\]

(4)

\[
\dot{x} = f(x, u, t), \quad t \in [t_0, t_f],
\]

(5)

\[
x^0 = f^0(x^0, u^0, \eta), \quad \eta \in [\tau, t_f], \quad x(t) = x^0(t),
\]

\[(6)\]

\[
\tau \in [t_i, t_{i+1}] \subset [t_0, t_f], \quad \varphi^0(x(t_0), t_0) = 0,
\]

\[
\varphi^0(x(t_f), t_f) = 0, \quad \varphi^0(x^0(t_f), t_f) = 0,
\]

\[
x \in E^n, \quad x^0 \in E^n, \quad u \in \Omega \in E^m, \quad u^0 \in \Omega_0 \subset E^{m_0},
\]

where with respect to \( S(\cdot), \varphi^0(\cdot), \varphi^f(\cdot), \Phi(\cdot), f(\cdot), f^0(\cdot), \partial f / \partial x, \partial \Phi^0 / \partial x^0 \) the assumptions are performed that are similar to those assumptions made for corresponding functions of the problem (1)-(3); \( t_f \) – time moment at which system (5) reached finite variety \( Q_f = \left\{ (x(t_j), t_f) : \varphi^f(x(t_j), t_f) = 0 \right\} \) provided that the change of the dynamics of its movement in time interval \([t_i, t_{i+1}]\) did not take place; \( t^i_f \) is the time moment of finite variety movement \( Q_{f,i} = \left\{ (x^i(t^i_f), t^i_f) : \varphi^f(x^i(t^i_f), t^i_f) = 0 \right\} \) of system (6). In physical meaning the problem (4) – (6) may be interpreted as a problem of dynamic system optimal trajectory construction with probable malfunctions in time interval \([t_i, t_{i+1}]\) resulting in this system movement dynamics change. With this despite probable malfunctions the dynamic system should accomplish its main and auxiliary task [1]–[4]. Assuming that the system dynamics hasn’t been subjected to changes at time interval \([t_i, t_{i+1}]\) or that it could be changed at any time moment \( t \in [t_i, t_{i+1}] \) we come to the next auxiliary problem of vector optimization [7]–[10]

\[
I^p = \text{col}[I_0 \rightarrow \min, I_1 \rightarrow \min, \ldots, I_N \rightarrow \min],
\]

(7)

\[
I_0 = S(x(t_0), t_0); x(t_1), t_1; x(t_2), t_2; \ldots; x(t_N), t_N; x(t_f), t_f)
\]

\[
x(t_f), t_f \] \[ + \int_{t_0}^{t_f} \Phi(x, u, t) dt,
\]

(8)

\[
\dot{x} = f(x, u, t), \quad t \in [t_0, t_f],
\]

(9)

\[
\varphi^0(x(t_0), t_0) = 0, \quad \varphi^f(x(t_f), t_f) = 0,
\]

(10)

\[
x \in E^n, \quad x^0 \in E^n, \quad u \in \Omega \in E^m, \quad u^0 \in \Omega_0 \subset E^{m_0},
\]

\[
x(t_i) = x^0(t_i), \quad (i = 1, N), \quad t_1 < t_2 < \ldots < t_{N-1} < t_N,
\]

where \( t^i_f \) is the time moment at which finite variety \( Q_{f,i} \) is reached by system (9), that began its movement at time moment \( t_i \), and this problem differs from the problem (4) – (6) by that fact that the condition allowing infinitely large number of time moments of system dynamics change was substituted by less rigid condition assuming that a finite number of points \( t^i(i = 1, N) \) of system dynamics change. As the requirements (7) – (10) are less rigid than (4) – (6) each feasible process \( x(t), u(t), \varphi^0(\eta), u^0(\eta), t_0, t_f, t_1, t_{N} \) of the problem (4) – (6) will be admissible as well as in the problem (7) – (10) at arbitrary chosen \( t_i(i = \overline{2, N-1}) \) and \( t_{i+1} < t_i(i = \overline{1, N}) \). We shall introduce the notion of steady optimal process and establish its features as in [5], [8].

Process \( x(t), u(t), \varphi^0(\eta), u^0(\eta), t_0, t_f, t_1, t_{N} \) optimal for the problem (4) – (6) is considered as steady optimal if natural \( N_0 \) exists and it is so that for countable variety of values \( N > N_0 \) the admissible process \( x(\tau), u(\tau), \varphi^0(\eta), u^0(\eta), T, t_0, t_f \), having vector \( T = (t_1, t_2, \ldots, t_{N-1}, t_N) \) consisting of fixed values \( t_1 < t_2 < \ldots < t_{N-1} < t_N \) is also optimal in problem (7) – (10). We recognize that \( N > N_0 \) and pass to the branched trajectory optimization problem using the principle of quasi-branching of system trajectory (9) at time moments \( t(i = \overline{1, N}) \) changing from trajectory of system (8), moving from original variety \( Q_0 = \left\{ (x(t_0), t_0) : \varphi^0(x(t_0), t_0) = 0 \right\} \) to finite variety \( Q_f \). This principle is based on the following considerations.
Suppose that system dynamics (8) will change at time moment \( \tau = t_i \). In this case dynamic system trajectory should consist of two sections optimally joint together by condition of stop change [4]. In the interval \([t_0, t_1]\) trajectory is given by equation (8), and in the interval \([t_1, t_f]\) – by equation (9).

Suppose that for reaching time moment \( t_i \) system dynamics change didn’t occur. Then system movement trajectory continues to be described by equation (8) until the next hypothetical moment \( t_2 \) of its dynamics change and change-over to the system moment trajectory description by equation (9). However trajectory section in time interval \([t_1, t_2]\) should be optimally joint at time moment \( t_i \) with two above mentioned sections. Hence it follows that at time moment \( t_i \) it is necessary to observe the condition of stop change for systems moving on branched trajectories [4] but not for disintegrated systems [2], [5].

Viewing in similar way the condition of trajectory step change at time moments \( t_2, t_3 \) and so on until \( t_N \) we come to the problem of optimization of branched trajectory with criterion of

\[
\dot{t} = v^N \left( \Phi(x, u, t)dt \right) + \sum_{j=1}^{N} \mu^N_j \int_{t_j}^{t_f} \Phi^0(x^0, u^0, \eta)d\eta + \xi^{N,0}(x^0(t_j), t_j).
\]

(11)

According to [4] for process optimum \( \tilde{x}(t), \tilde{u}(t), \tilde{x}^0(\eta), \tilde{u}^0(\eta) \), \( T, t_0, \tilde{t} \), the problems (8) – (11) there are solutions \( \lambda^N(t), \lambda^0^N(\eta) \) of adjoint vector equations

\[
\begin{align*}
\lambda^N(t) + \frac{\partial H(\tilde{x}(t), \tilde{u}(t), \lambda^N(t), t)}{\partial \tilde{x}(t)} = 0, \\
\mu^N \left[ \lambda^0^N(\eta) + \frac{\partial H^0(\tilde{x}^0(\eta), \tilde{u}^0(\eta), \lambda^0^N(\eta), \eta)}{\partial \tilde{x}^0(\eta)} \right] = 0,
\end{align*}
\]

(12)

such that the conditions following beneath are valid:

(1°) of transversality

\[
\begin{align*}
v^N \left[ \frac{\partial S}{\partial \tilde{x}(t_0)} \right] + \lambda^N(t_0) + \frac{\partial \phi^{0,0}(\tilde{x}(t_0), t_0)}{\partial \tilde{x}(t_0)} \xi^{N,0} = 0,
\end{align*}
\]

(13)

(2°) of step change

\[
\begin{align*}
v^N \left[ \frac{\partial S}{\partial \tilde{x}(t_i)} \right] + \lambda^N(t_i - 0) - \lambda^N(t_i - 0) + \mu^N \lambda^0^N(t_i) = 0, \\
v^N \left[ \frac{\partial S}{\partial \tilde{x}(t_f)} \right] + \lambda^N(t_f - 0) - \lambda^N(t_f - 0) + \mu^N \lambda^0^N(t_f) = 0,
\end{align*}
\]

(20)

(3°) of Hamiltonian minimum

\[
H(\tilde{x}(t), \tilde{u}(t), \lambda^N(t), t) = \min_{u(t) \in \Omega_u} H(\tilde{x}(t), u(t), \lambda^N(t), t),
\]

(21)

\[
H^0(\tilde{x}^0(\eta), \tilde{u}^0(\eta), \lambda^0^N(\eta), \eta) = \min_{u^0(\eta) \in \Omega_{u^0}} H^0(\tilde{x}^0(\eta), u^0(\eta), \lambda^0^N(\eta), \eta),
\]

(22)

(4°) of nontriviality and nonnegativity:

\[
\begin{align*}
\sum_{j=1}^{N} \mu^N_j = 1, \sum_{j=1}^{N} \tilde{x}^{N}_j + \sum_{j=1}^{N} \tilde{x}^{N}_j = 1, \\
\tilde{x}^{N}_j \geq 0 (j = 1, r^{0,0}), \tilde{x}^{0,0}_j \geq 0, \quad (j = 1, r^{0,0}), \quad \mu^N_j \geq 0,
\end{align*}
\]

(23)

\[
\begin{align*}
\xi^{N}_0 \geq 0 (j = 1, r^{0,0}), \quad \xi^{N}_j \geq 0, \quad (j = 1, r^{0,0}), \quad \mu^N_j \geq 0,
\end{align*}
\]

(24)
Here marks "v" means optimum variables and parameters:

\[
H(\cdot) = \Phi \left( x(t), u(t) + \lambda^N_T(t) f(x(t), u(t), t) \right),
\]

\[
H^N(\cdot) = \Phi^N \left( x^N(\eta), u^N(\eta), \eta \right)
\]

+ \lambda^N_T(\eta) f^N \left( x^N(\eta), u^N(\eta), \eta \right)

Let’s set on numerical axis step functions

\[
\mu^N(\tau) \equiv \xi^N_j(\tau) \quad (j = 1, r^{(f)}_N)
\]

with step change correspondingly \( \mu^N_1, ..., \mu^N_N, \; \xi^N_{j1}, ..., \xi^N_{jN} \) at points \( t_1, ..., t_N \). At \( \tau < t_1 \) assume that \( \mu^N(\tau) = \xi^N_j(\tau) = 0 \).

In time interval \([t_1, t_N]\) the adjoint equation (14) together with the condition of step change (20) is written in equivalent integral form including Stieltjes integral,

\[
\lambda^N(\tau) = v^N \int^\tau \frac{\partial H \left( \hat{x}(t), \hat{u}(t), \lambda(t), t \right)}{\partial x(t)} dt
\]

\[+ \int^\tau \lambda^N(t) d\mu^N(t) + v^N \left[ \frac{\partial S}{\partial x(t)} \bigg|_{t_N} + \lambda^N(t_N + 0) \right],
\]

where \( \tau \in [t_1, t_N] \), \( \lambda^N(t_N + 0) \) – in the result of solution of equation (12) is in the interval of \([t_N, t_f]\) observance of limiting conditions (18). In this case the condition of step change (19) will take the form

\[
v^N \left[ \frac{\partial S}{\partial x(t)} \bigg|_{t_N} \right] \lambda^N(t_N + 0) - \lambda^N(t_N - 0) = 0,
\]

where \( \lambda^N(t_N - 0) \) – is obtained as a result of the solution of equation (12) in the interval \([t_0, t_1]\) limiting condition (13).

Due to the condition (4’s) functions \( \mu^N(\tau), \xi^N_j(\tau) \) are nonnegative, bounded and have bounded variation. Take arbitrary section \( J \) of numerical axis including \([t_1, t_N]\) together with small neighborhood and choose \( N > N_0 \) such, that \( t_1, ..., t_N \) remained to be the points of continuity of control \( u(t) \). From the sequence of functions \( \{ \mu^N(t) : N > N_0 \} \), \( \{ \xi^N_j(t) : N > N_0 \} \) one can isolate subsequences that by points on \( J \) converge to limiting functions \( \mu(t), \xi^N_j(t) \) that is

\[\mu^N(t) \rightarrow \mu(t), \; \xi^N_j(t) \rightarrow \xi^N_j(t), \; \tau \in J .\]

Scalar \( v^N \) and vectors \( \xi^N_0, \xi^N_f, \; N > N_0 \) are also bounded in a set and therefore the sequences \( \{ v^N : N > N_0 \}, \; \{ \xi^N_0 : N > N_0 \}, \; \{ \xi^N_f : N > N_0 \} \) have convergent subsequences, that is \( v^N \rightarrow v, \; \xi^N_0 \rightarrow \xi_0, \; \xi^N_f \rightarrow \xi_f \). Variations of functions \( \mu(t) \) on \( J \) are bounded, hence in integral equation of Voltaire type

\[
\lambda(\tau) = v \left[ \frac{\partial S}{\partial x(t)} \bigg|_{t_N} + \lambda(t_N + 0) \right]
\]

\[+ v \int^{t_N} \frac{\partial H}{\partial x} dt + \int^{t_N} \lambda^N(t) d\mu(t),
\]

the last summand has a meaning. The solution \( \lambda(\tau) \) of this equation exists in a class of functions of bounded variation and therewith it is sole [9].

Due to (27) the solutions \( \lambda^N(\tau) \) of equation (25) at each point \( \tau \in J \) come to the solution \( \lambda(\tau) \) of equation (28). With this function \( \mu(t) \) as a limit (4’s) of nondecreasing nonnegative function is itself nondecreasing nonnegative function on \( J \) and hence it can be considered as a measure.

Passing to the limit of change to \( N(N \rightarrow \infty, \max(t_{N+1} - t_N) \rightarrow 0) \) in relationships (12) – (18), (21) – (24), (26), taking into consideration all above we obtain the following result.

Let \( \hat{x}(t), \hat{u}(t), \hat{x}^N(\eta), \hat{u}(\eta), \hat{t}_0, \hat{t}_f, \hat{t}, \hat{t}_N \) too be a stable optimal process of problem (4) – (6).

In this case there are nonnegative numbers \( \xi_{j0} \left( j = 1, r^{(f)}_N \right), \xi_{jf} \left( j = 1, r^{(f)}_N \right), \; v \) and nonnegative measures \( \mu(t), \xi_j(t) \left( j = 1, r^{(f)}_N \right) \) of a bounded function that are concentrated on variety \( M = \{ t : t \in [t_1, t_N] \} \), there is vector function \( \lambda(\tau) \) of bounded variation being the solution of integral equation (28) for \( t \in [t_1, t_N] \) and of conventional differential equation

\[
\lambda(t) = - \frac{\partial H}{\partial x}, \quad \text{for } t \in [t_0, t_f] \subset [t_1, t_N]
\]

and there is a vector function \( \lambda^0(\eta) \) of bounded variation being the solution of equation

\[
\lambda^0(\eta) = - \frac{\partial H^0}{\partial x}, \quad \eta \in [\tau, t_f],
\]

such that the following conditions are valid:
(1°) of transversality

\[
\left[ \frac{\partial S}{\partial \xi(t_0)} + \lambda (\dot{i}_0) \right] + \left[ \frac{\partial (\psi(\xi(t_0)))}{\partial \xi(t_0)} \right] = 0,
\]

\[
\left[ \frac{\partial S}{\partial \xi(t_f)} - H(\dot{i}_f), \dot{u}(\dot{i}_f), \lambda(\dot{i}_f), \dot{i}_f \right] + \left[ \frac{\partial (\psi(\xi(t_f)))}{\partial \xi(t_f)} \right] = 0,
\]

\[
\mu(\tau)\lambda (\dot{i}_f) + \left[ \frac{\partial (\psi(\xi(t_f)))}{\partial \xi(t_f)} \right] d\xi(\tau) = 0,
\]

\[
\left[ \frac{\partial S}{\partial \xi(t_f)} + H(\dot{i}_f), \dot{u}(\dot{i}_f), \lambda(\dot{i}_f), \dot{i}_f, \dot{i}_f \right] + \left[ \frac{\partial (\psi(\xi(t_f)))}{\partial \xi(t_f)} \right] = 0.
\]

(2°) of step change

\[
\frac{\partial S}{\partial \xi(t_i)} + \lambda(t_i + 0) - \lambda(t_i - 0) = 0.
\]

(3°) of Hamiltonian minimum

\[
H(\dot{i}(t), \dot{u}(t), \lambda(t), t) = \min_{u(t) \in \mathbb{R}} H(\dot{i}(t), u(t), \lambda(t), t),
\]

\[
H^0(\dot{x}(\eta), \dot{u}(\eta), \lambda(\eta), \eta) = \min_{x(\eta) \in \mathbb{R}} H^0(\dot{x}(\eta), u(\eta), \lambda(\eta), \eta).
\]

(4°) of nontriviality

\[
v \int_{t_i}^{t_f} d\mu(t) = 1, \sum_{j=1}^{J_0} \xi_{j0} + \sum_{j=1}^{J_1} \xi_{j0} + \sum_{j=1}^{J_2} \xi(0)(t) = 1.
\]

IV. CONCLUSION

The dynamic system’s path complying with stated above conditions has the feature – it provides to dynamic system the additional abilities to proceed on new paths during fixed time. These paths allow performing auxiliary (additional/alternative) task in case of system’s dynamics changes due to faults and damages as well in case of on-line retargeting.

REFERENCES


Received September 16, 2016

Lysenko Alexander. Doctor of Engineering Science. Professor
Professor of Department of Telecommunications, National Technical University of Ukraine
«Igor Sikorsky Kyiv Polytechnic Institute», Kyiv, Ukraine.
Research interests: the dynamics of flight control systems.
Publications: more than 300 papers.
E-mail: lysenko.a.i.1952@gmail.com

Tachinina Helen. Candidate of Science (Engineering). Associate Professor.
Associate professor, Department of Automation and Energy Management, National Aviation University.
Education: Kyiv International University of Civil Aviation, Kyiv, Ukraine (1999).
Research area: The methods of optimal control of compound dynamic systems
Publications: 85
E-mail: tachinina@rambler.ru
О. І. Лисенко, О. М. Тачиніна. Принцип оптимальності для траєкторії виведення динамічної системи з альтернативою
Розглянуто задачу Лагранжа з урахуванням дії функціонального обмеження в кожен момент часу на заданому інтервалі. Отримано необхідні умови оптимальності траєкторії детермінованої динамічної системи, що інтерпретується як траєкторія виведення космічного літального апарату.
Ключові слова: оптимізація; динамічна система; розгалужені траєкторії; оптимальна траєкторія.

Лисенко Олександр Іванович. Доктор технічних наук. Професор.
Посада, місце роботи: професор кафедри телекомуникацій, Національний Технічний Університет України «Київський політехнічний інститут ім. І. Сікорського»
Напрями наукової діяльності: динаміка польоту, системи керування.
Кількість публікацій: більше 300 наукових робіт.
E-mail: lysenko.a.i.1952@gmail.com

Тачиніна Олена Миколаївна. Кандидат технічних наук. Доцент.
Посада, місце роботи: доцент, кафедра автоматизації та енергоменеджменту, Національний Авіаційний Університет.
Освіта: Київський міжнародний університет цивільної авіації, Київ, Україна (1999).
Напрями наукової діяльності: методи оптимального керування складеними динамічними системами.
Кількість публікацій: 85
E-mail: tachinina@mail.ru